Math 244 worksheet on differentiability

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First read Section 14.8, a few times if necessary.

Recall from Calculus I the following definition:

Definition 1. A function $f(x)$ is differentiable at x_0 if

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists.

Another way to characterize differentiability is the following definition.

Definition 2. A function $f(x)$ is differentiable at x_0 if there is a linear function $L(x) = m(x - x_0) + f(x_0)$ that is a good approximation to f in the sense that the error $E(x) = f(x) - L(x)$ satisfies

$$\lim_{x \to x_0} \frac{E(x)}{x - x_0} = 0.$$

The idea here is that $L(x)$ is a good approximation to $f(x)$ if the error goes to 0 as $x \to x_0$ faster than $x - x_0$ goes to 0.

1. Show that if $f(x)$ satisfies Definition 2 then

   $$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

   exists and $m = f'(x_0)$.

   Compare your proof with the one given in Example 1 in Section 14.8.

2. Now, show that if $f(x)$ satisfies Definition 1 then $L(x) = f'(x_0)(x - x_0) + f(x_0)$ is a linear approximation to f such that the error $E(x) = f(x) - L(x)$ satisfies

   $$\lim_{x \to x_0} \frac{E(x)}{x - x_0} = 0.$$

   Note that this says if $f(x)$ satisfies Definition 1 then it must also satisfy Definition 2. In part 1, you proved that if $f(x)$ satisfies Definition 2 then it must also satisfy Definition 1. Therefore the two definitions are logically equivalent.

3. Recall that $f(x)$ is continuous at x_0 if

   $$\lim_{x \to x_0} f(x) = f(x_0).$$

   Theorem 2.1 in Section 2.6 in your textbook says that if $f(x)$ satisfies Definition 1 then $f(x)$ is continuous at x_0. Prove this (better) or review the proof in the book (not as good).

   Now prove that if $f(x)$ satisfies Definition 2 then $f(x)$ must be continuous at x_0.

4. Definition 1 does not readily extend to the multivariable setting. Requiring that all of the partial derivatives exist is not enough to guarantee the usual nice behaviors that we expect from differentiable functions. The definition of differentiability in Section 14.8 is the generalization of Definition 2 to the multivariable setting.

   Prove that if $f(x, y)$ is differentiable at the point (x_0, y_0), then it must be continuous there. (Hint: try to generalize your proof from part 3 to the multivariable setting.)

5. We proved in class that if $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable and $\vec{u}$ is a unit vector, then

$$f_{\vec{u}}(x_1, \ldots, x_n) = \vec{\nabla} f \cdot \vec{u}.$$

The multivariable chain rule by a very similar argument. Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $x = x(t)$, $y = y(t)$ all be differentiable functions. We want to prove that

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

By definition,

$$f'(t_0) = \lim_{t \to t_0} \frac{f(x(t), y(t)) - f(x(t_0), y(t_0))}{t - t_0}.$$

For the sake of simpler notation, let $x = x(t)$, $y = y(t)$, $x_0 = x(t_0)$, and $y_0 = y(t_0)$. Use the differentiability of f to write $f(x, y) = L(x, y) + E(x, y)$ in the above difference quotient. To show that

$$\lim_{t \to t_0} \frac{E(x, y)}{t - t_0} = 0$$

write

$$\frac{E(x, y)}{t - t_0} = \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} \frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{t - t_0}$$

and think about what happens to

$$\frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{t - t_0}$$

as $t \to t_0$. To deal with sign issues, consider the one-sided limits as $t \to t_0^+$ and as $t \to t_0^-$ separately. Complete the details of this proof.