Math 302 Exam 1 Solutions

Oct 6, 2010

1. (10 pts) Write a derivation for the following argument.

   "Each high school student in Slumpville who takes an honors class is cool. There is a high school student in Slumpville who is smart and not cool. Therefore there is a high school student in Slumpville who is smart and not taking an honors class."

   Let

$$\begin{aligned} S &= \text{set of high school students in Slumpville} \\ P(x) &= x \text{ is taking an honors class} \\ Q(x) &= x \text{ is cool} \\ R(x) &= x \text{ is smart} \end{aligned}$$

   Then the argument above is

$$\begin{array}{l} (\forall\ x \text{ in } S)[P(x) \rightarrow Q(x)] \\ (\exists\ x \text{ in } S)[R(x) \wedge \neg Q(x)] \\ \hline (\exists\ x \text{ in } S)[R(x) \wedge \neg P(x))] \end{array}$$

   And its derivation is

$$\begin{array}{ll} (1) & (\forall\ x \text{ in } S)[P(x) \rightarrow Q(x)] \\ (2) & (\exists\ x \text{ in } S)[R(x) \wedge \neg Q(x)] \\ \hline (3) & R(a) \wedge \neg Q(a) \\ (4) & P(a) \rightarrow Q(a) \\ (5) & \neg Q(a) \\ (6) & \neg P(a) \\ (7) & R(a) \\ (8) & R(a) \wedge \neg P(a) \\ (9) & (\exists\ x \text{ in } S)[R(x) \wedge \neg P(x))] \end{array} \qquad \begin{array}{l} \\ \\ \text{(2), existential instantiation} \\ \text{(1), universal instantiation} \\ \text{(3), simplification} \\ \text{(4),(5), modus tollens} \\ \text{(3), simplification} \\ \text{(6),(7) adjunction} \\ \text{(8), existential generalization} \end{array}$$

2. (5 pts each) Negate the following statements without using phrases like "it is not the case" or "it is not true"

   (a) Every house has a door that is white.

   Let S be the set of all houses, $T(x)$ the set of doors of house x, and $P(y) = y$ is white. Then the above sentence has the structure

$$(\forall x \in S)(\exists y \in T(x))P(y).$$

   So its negation is

$$\neg(\forall x \in S)(\exists y \in T(x))P(y) \iff (\exists x \in S)(\forall y \in T(x))\neg P(y).$$

   That is "there is a house such that all of its doors are not white" or a little more naturally "there is a house none of whose doors are white."

   (b) At least one person in New York City owns every book published in 1990.

   Let S be the set of all people in NYC, T the set of books published in 1990, and $P(x,y) = x$ owns y. Then the above sentence has the structure

$$(\exists x \in S)(\forall x \in T)P(x,y).$$

So its negation is

$$\neg(\exists x \in S)(\forall x \in T)P(x,y) \iff (\forall x \in S)(\exists x \in T)\neg P(x,y).$$

That is "for every person in NYC there is a book published in 1990 that this person does not own." Of course, it is more natural to say "nobody in NYC owns every book published in 1990."

3. (5 pts each) Let P and Q be statements. Prove the following
   (a) $(P \to Q) \wedge \neg Q \implies P$

   This one is not actually true, as we will see from the truth table:

$((P$	$\to$	$Q)$	$\wedge$	$\neg$	$Q)$	$\to$	P
T	T	T	F	F		T	
T	F	F	F	T		T	
F	T	T	F	F		T	
F	T	F	T	T		F	
1	3	2	5	4		**6**	

   Remark: So this one was meant to be modus tollens, except for the typo that the $\neg$ was missing. Since this was my fault, I of course did not hold it against you. Most of you constructed the correct truth table and found that this was not actually an implication. If your truth table was almost correct except you tried desperately to prove what was not true, I also gave you full credit.

   (b) $\neg(P \vee Q) \iff \neg P \wedge \neg Q$

$(\neg$	$(P$	$\vee$	$Q))$	$\leftrightarrow$	$(\neg$	P	$\wedge$	$\neg$	$Q)$
F	T	T	T	T	F		F	F	
F	T	T	F	T	F		F	T	
F	F	T	T	T	T		F	F	
T	F	F	F	T	T		T	T	
4	1	3	2	**8**	5		7	6	

4. (a) (2 pts each) State the converse and the contrapositive of the statement "If a quadrilateral has four equal sides then its diagonals are perpendicular."

   Converse: If a quadrilateral's diagonals are perpendicular then it has four equal sides. Contrapositive: If a quadrilateral's diagonals are not perpendicular then it does not have four equal sides.

   (b) (3 pts) Prove that a conditional statement is equivalent to its contrapositive.

   The truth table below shows that $(P \to Q) \leftrightarrow (\neg Q \to \neg P)$ is a tautology:

$(P$	$\to$	$Q)$	$\leftrightarrow$	$(\neg$	Q	$\to$	$\neg$	$P)$
T	T	T	T	F		T	F	
T	F	F	T	T		F	F	
F	T	T	T	F		T	T	
F	T	F	T	T		T	T	
1	3	2	**8**	5		7	6	

(c) (3 pts) Prove that a conditional statement is not equivalent to its converse.

The truth table below shows that $(P \to Q) \leftrightarrow (Q \to P)$ is not a tautology:

$(P$	$\to$	$Q)$	$\leftrightarrow$	Q	$\to$	$P)$
T	T	T	T		T	
T	F	F	F		F	
F	T	T	F		T	
F	T	F	T		T	
1	3	2	**6**		5	

5. The following are the premises of an argument:

"Turtledoves are either birds or reptiles. If turtledoves have feathers than they are birds. It is not the case that if turtledoves have scales then they are reptiles. Turtledoves are animals or they cannot fly."

(a) (3 pts) Add a statement to these premises to make them inconsistent.

We can just add the negation of the very first statement: "Turtledoves are neither birds nor reptiles." Then the first statement and this statement already form a contradiction, so the premises cannot be consistent.

(b) (7 pts) Use these premises (the original ones + yours) to derive the conclusion "Turtledoves can do cube roots in their heads."

Let

$$
\begin{aligned}
P &= \text{Turtledoves are birds.} \\
Q &= \text{Turtledoves are reptiles.} \\
R &= \text{Turtledoves have feathers.} \\
S &= \text{Turtledoves have scales.} \\
T &= \text{Turtledoves have animals.} \\
U &= \text{Turtledoves can fly.} \\
V &= \text{Turtledoves can do cube roots in their heads.}
\end{aligned}
$$

Here is the argument with my premise as (5):

$$
\begin{array}{ll}
(1) & P \vee Q \\
(2) & R \to P \\
(3) & \neg(S \to Q) \\
(4) & T \vee \neg U \\
(5) & \neg P \wedge \neg Q \\
\hline
 & V
\end{array}
$$

And here is the derivation:

(1)	$P \vee Q$	
(2)	$R \rightarrow P$	
(3)	$\neg(S \rightarrow Q)$	
(4)	$T \vee \neg U$	
(5)	$\neg P \wedge \neg Q$	
(6)	$\neg Q$	(5), simplification
(7)	$\neg P$	(5), simplification
(8)	P	(1), (6), modus tollendo ponens
(9)	$P \vee V$	(8), addition
(10)	V	(7), (9), modus tollendo ponens

6. (5 pts each) Translate the following into Block World syntax.
   (a) Every pentagon has a triangle in the same row which is bigger than it.

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A x (Pentagon(x) => (E y (Triangle (y) /\ SameRow(x,y) /\ Smaller(x,y))
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   (b) There is a unique medium square on the board.

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E x (Medium(x) /\ Square(x) /\ (A y ((Medium(y) /\ Square(y)) => y=x))
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7. (10 pts) **Extra credit problem.** On one of your explorations in Logica, you go treasure hunting in the northwestern corner of the country. The people who live there all understand English just fine, but because of a strange historic accident–which no one quite remembers–instead of yes and no, they say *boo* and *bah*. Unfortunately, we don't know which of these means yes and which means no. Once again, you come across a two-door-situation: one door leads to the treasure, the other to a bloodthirsty monster. One guard is guarding the doors. He is either a square shooter or a liar. He will answer one boo/bah question. What question do you ask to find out which door leads to the treasure?

   One such question is "Would you answer boo if I asked you whether the door on your right leads to the treasure?" Then if the answer is boo, go through the door to the guard's right, if the answer is bah, go through the door to the guard's left. For example, if the guard is a liar, the door to the treasure is on his right, and boo means no, then the guard would actually answer boo if asked whether the door on your right leads to the treasure, but he would have to lie about what he would answer, so he will deny that he would answer boo. To deny, he will have to say boo. That remaining cases are summarized in the table below:

the guard is a	the door with the treasure is on the guard's	boo means	the guard's answer
square shooter	right	yes	boo
square shooter	right	no	boo
square shooter	left	yes	bah
square shooter	left	no	bah
liar	right	yes	boo
liar	right	no	boo
liar	left	yes	bah
liar	left	no	bah