Math 302 Exam 2 Solutions

Nov 10, 2010

1. (10 pts) Let A, B, and C be sets. Prove that
$$(A \cap B) \cap C = A \cap (B \cap C).$$

Notice that
$$\begin{aligned} x \in (A \cap B) \cap C &\iff x \in A \cap B \text{ and } x \in C \\ &\iff x \in A, x \in B, x \in C \\ &\iff x \in A \text{ and } x \in B \cap C \\ &\iff x \in A \cap (B \cap C) \end{aligned}$$

That is $x \in x \in (A \cap B) \cap C$ if and only if $x \in A \cap (B \cap C)$. So $(A \cap B) \cap C = A \cap (B \cap C)$.

2. (5 pts each) Let x and y be real numbers. Prove the following statements.

(a) $|x - y| = |y - x|$.

First, observe that if $z \geq 0$ then $|z| = z$, and if $z \leq 0$ then $|z| = -z$. We will consider two cases:

Case $x - y \geq 0$:
$$\begin{aligned} |x - y| &= x - y \\ |y - x| &= -(y - x) = x - y \end{aligned}$$

So $|x - y| = |y - x|$.

Case $x - y < 0$:
$$\begin{aligned} |x - y| &= -(x - y) = y - x \\ |y - x| &= y - x \end{aligned}$$

So $|x - y| = |y - x|$.

(b) $|xy| = |x||y|$.

This is similar, but now we have four cases to consider.

Case $x \geq 0$ and $y \geq 0$: Then $xy \geq 0$. So $|xy| = xy$. Also $|x| = x$ and $|y| = y$. That is $|xy| = xy = |x||y|$.

Case $x < 0$ and $y < 0$: Then $xy > 0$. So $|xy| = xy$. Also $|x| = -x$ and $|y| = -y$. That is $|xy| = xy = (-x)(-y) = |x||y|$.

Case $x \geq 0$ and $y < 0$: Then $xy \leq 0$. So $|xy| = -xy$. Also $|x| = x$ and $|y| = -y$. That is $|xy| = -xy = x(-y) = |x||y|$.

Case $x < 0$ and $y \geq 0$: Then $xy \leq 0$. So $|xy| = -xy$. Also $|x| = -x$ and $|y| = y$. That is $|xy| = -xy = (-x)y = |x||y|$.

3. (5 pts each) Prove or give a counterexample to each of the following statements.

(a) For each integer a, there exists an integer b such that $a|b$.

This is true. Let a be any integer. Set $b = a$. Then $b = 1a$, so $a|b$.

(b) There exists an integer b such that for all integers a, we have $a|b$.

This is also true. Let a be any integer. Then $0 = 0a$, so $a|0$. Hence $b = 0$ is such an integer.

Remark: Notice that the statement in part (b) implies the statement in part (a). So we could have proved part (a) by simply referring to part (b). I spelled out a different proof for part (a) so you can see the difference between the two statements: in part (a), b can depend on a; in part (b), b must work universally for all $a \in \mathbb{Z}$.

4. (a) (2 pts) State the definition of a prime number.

   See definition on p. 72 of your textbook.

   (b) (8 pts) Prove that there are infinitely many prime numbers.

   See Theorem 2.3.4 in your textbook.

5. (a) (2 pts) Let A be a set. State the definition of the power set $\mathcal{P}(A)$.

$$\mathcal{P}(A) = \{X \mid X \subseteq A\}.$$

   Or see definition on p. 115 of your textbook.

   (b) (8 pts) Let A be a finite set of n elements. How many elements does $\mathcal{P}(A)$ have? Don't forget to justify your answer.

   $\mathcal{P}(A)$ has 2^n elements. To construct any subset X of A, you can go through the elements of A, and decide for each element whether to include it in X. That is two choices for each element. There are n elements. That gives

$$\underbrace{2 \cdot 2 \cdots 2}_{n \text{ factors}} = 2^n$$

   different subsets.

6. (a) (2 pts) State the definition of rational number.
   See definition on p. 70 in your textbook.

   (b) (4 pts) Prove that if x and y are rational numbers then xy is a rational number.

   Let $x, y \in \mathbb{Q}$. So $x = m/n$ and $y = s/t$ for some $m, n, s, t \in \mathbb{Z}$ such that $n, t \neq 0$. Now

$$xy = \frac{m}{n}\frac{s}{t} = \frac{ms}{nt}.$$

   The integers are closed under multiplication, so $ms, nt \in \mathbb{Z}$. Also $n, t \neq 0 \implies nt \neq 0$. Hence $xy \in \mathbb{Q}$.

   (c) (4 pts) Prove that if x and y are rational numbers then $x + y$ is a rational number.

   Let $x, y \in \mathbb{Q}$. So $x = m/n$ and $y = s/t$ for some $m, n, s, t \in \mathbb{Z}$ such that $n, t \neq 0$. Now

$$x + y = \frac{m}{n} + \frac{s}{t} = \frac{mt + sn}{nt}.$$

   The integers are closed under multiplication and addition, so $mt + sn \in \mathbb{Z}$ and $nt \in \mathbb{Z}$. Also $n, t \neq 0 \implies nt \neq 0$. Hence $xy \in \mathbb{Q}$.

7. (10 pts) **Extra credit problem.** Determine the number of different triples of sets A, B, C such that

(a) $A \cup B \cup C = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
(b) $A \cap B \cap C = \emptyset$
(Hint: a Venn diagram may be helpful.)

A priori, if x is an integer between 1 and 10 then $x \in A$ or $x \notin A$. Also $x \in B$ or $x \notin B$, and $x \in C$ or $x \notin C$. That is eight different possibilities. The first condition tells us that $x \notin A, B, C$ is not a possibility. The second condition tells us that $x \in A, B, C$ is not a possibility. That leaves six possible places to put each of the ten numbers, yielding 6^{10} different outcomes. So there are 6^{10} different triples of such sets.

You can easily see this in the Venn diagram below, where I numbered the six possible places.