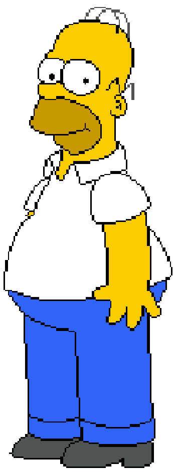
Math 302 Exam 2 Solutions

Apr 11, 2007

1. Consider the following argument.

   If Homer Simpson is fat or short then he does not like beer. If Homer hangs out at Moe's then likes beer. Homer is short and hangs out at Moe's. Therefore if Homer is short then he is fat.

   (a) (5 pts) Are the premises consistent? Explain why or why not. If you use symbols to represent statements, make sure you explain what each symbol means.

   Let

   $$P = \text{Homer is fat.}$$
   $$Q = \text{Homer is short.}$$
   $$R = \text{Homer likes beer.}$$
   $$S = \text{Homer hangs out at Moe's.}$$

   Then the argument is

   $$\begin{array}{l} (P \vee Q) \implies \neg R \\ S \implies R \\ Q \wedge S \\ \hline Q \implies P \end{array}$$

   For $Q \wedge S$ to be true, both Q and S must be true. Then R must be true otherwise $S \implies R$ would be false. Then $\neg R$ is false. Since Q is true, $P \vee Q$ is also true, so $(P \vee Q) \implies \neg R$ is then false. This shows there is no way to make all three premises true, hence the premises are inconsistent.

   (b) (10 pts) If the argument is valid, give a derivation. If it is invalid, explain why it is invalid.

   The argument is valid since the premises are inconsistent. Any conclusion can be derived from inconsistent premises. Here is one possible derivation based on the same idea we used in class (and most of you used on the HW) to exploit the inconsistency in the premises.

(1) $(P \vee Q) \implies \neg R$	
(2) $S \implies R$	
(3) $Q \wedge S$	
(4) Q	(3), Simplification
(5) $P \vee Q$	(4), Addition
(6) $\neg R$	(1), (5), Modus Ponens
(7) S	(3), Simplification

(8) R	(2), (7), Modus Ponens
(9) $R \vee (Q \implies P)$	(8), Addition
(10) $Q \implies P$	(9), (6), Modus Tollendo Ponens

If adding $Q \implies P$ to R worries you, just call it U, do the addition and the Modus Tollendo Ponens with U and then finally substitute $Q \implies P$ for U.

2. (5 pts) Negate the following sentence without using the word *not*.
"For every $x \in G$ and $y \in H$ there exists a $z \in G$ such that $xy = zx$."
Carefully explain why your answer is indeed the negation of the original sentence. (Hint: you can use logic symbols.)

Let
$$P(x, y, z) = \text{It is the case that } xy = zx.$$
Then the original sentence is $\forall x \in G, \forall y \in H, \exists z \in G, P(x, y, z)$. The negation is
$$\begin{aligned} \neg(\forall x \in G, \forall y \in H, \exists z \in G, P(x,y,z)) &\iff \exists x \in G, \neg(\forall y \in H, \exists z \in G, P(x,y,z)) \\ &\iff \exists x \in G, \exists y \in H, \neg(\exists z \in G, P(x,y,z)) \\ &\iff \exists x \in G, \exists y \in H, \forall z \in G, \neg P(x,y,z) \end{aligned}$$
This is "There exist $x \in G$ and $y \in H$ such that for all $z \in G$, $xy \neq zx$."

Notice that "there exist $x \in G$ and $y \in H$ for all $z \in G$ such that $xy \neq zx$" is not the negation of the original sentence. This sentence is the same as "for all $z \in G$, there exist $x \in G$ and $y \in H$ such that $xy \neq zx$." This is a weaker statement than "there exist $x \in G$ and $y \in H$ such that for all $z \in G$, $xy \neq zx$," because it allows x and y to be different for each z.

3. (10 pts) Find the mistake in the following derivation and explain why it is a mistake.

(1) $(\forall x \in U)\ [P(x) \implies (Q(x) \vee R(x))]$
(2) $(\exists x \in U)\ [\neg P(x) \implies S(x)]$
(3) $(\forall x \in U)\ [\neg(R(x) \vee S(x))]$

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(4) $P(a) \implies (Q(a) \vee R(a))$	(1), Universal Instantiation
(5) $\neg P(a) \implies S(a)$	(2), Existential Instantiation
(6) $\neg(R(a) \vee S(a))$	(3), Universal Instantiation
(7) $\neg R(a) \wedge \neg S(a)$	(6), De Morgan's Law
(8) $\neg S(a)$	(7), Simplification
(9) $\neg\neg P(a)$	(5), (8), Modus Tollens
(10) $P(a)$	(9), Double Negation
(11) $Q(a) \vee R(a)$	(4), (10), Modus Ponens
(12) $\neg R(a)$	(7), Simplification
(13) $Q(a)$	(11), (12), Modus Tollendo Ponens
(14) $(\forall x \in U)\ Q(x)$	(13), Universal Generalization

The mistake is in the last line. Only an existential generalization could be done there because the existential instantiation in (5) restricts a to be some particular element of U.

Some of you said that the existential instantiation in line (5) needs a new letter, say b, because a already has a meaning. But the meaning a already has is that it is any element of U. Since b would still be an element of U, the previous meaning assigned to a does not interfere with this. Of course, once the existential instantiation is done, a is no longer any element of U, but some particular element of U. This is what causes the error later on. In fact, if we did use a new letter b in (4), we would find it difficult to do any logical manipulation with it later on because other things would be in terms of a.

4. (10 pts each) Prove the following theorems. You do not need to show scratch work, the cleaned-up proof will be sufficient.

   (a) Let k, m, n be integers. If $k|m$ and $k|m+n$ then $k|n$.

   Since $k|m$ and $k|m+n$, there are some integers p and q such that $m = pk$ and $m+n = qk$. Then
   $$n = (n+m) - m = qk - pk = (q-p)k.$$
   Since $q-p$ is the difference of two integers, it is also an integer. Hence $k|n$.

   (b) If x is an odd integer then integer division of x^2 by 4 gives a remainder of 1.

   If x is odd then $x = 2n+1$ for some $n \in \mathbb{Z}$. So
   $$x^2 = (2n+1)^2 = 4n^4 + 4n + 1 = 4(n^2+n) + 1.$$
   Since n is an integer, so is n^2 and then $n^2 + n$. Hence $4(n^2+n)$ is divisible by 4. The 1 in $4(n^2+n)+1$ gives the remainder of 1.

5. (10 pts) **Extra credit problem.**

   A young knight once entered the court of the old king. The king said to him "You are a fine young man and I am truly impressed with you. So I will marry one of my two daughters to you. To make this more interesting, I want you to make a statement. If the statement is true, then you will marry my older daughter. If it is false, you will marry my younger daughter." Unfortunately, both of the king's daughters were exceedingly mean and ugly. The knight had no interest in marrying either. What statement could he make to avoid having to marry either daughter? Justify your answer.

   As long as the knight makes a statement, it will either be true or false. So on first sight it looks like he is stuck with having to marry one of the daughters.

   But there is in fact a way out. He can make a statement which makes it impossible for the king to keep his word. Let us first assume that polygamy is against the law in this kingdom. Then the knight could say "you are not going to marry your older daughter to me." Notice that this is indeed a statement because ultimately, the king will either marry his older daughter to the knight or not. If the king indeed does not marry his older daughter to the knight, then the statement is true, so the king must marry his older daughter to the knight if he wants to keep his word. If the king does marry his older daughter to the knight, then the statement is false, so the king must marry his younger daughter to him if he wants to keep his word. But he cannot marry both daughters to the knight. Either way, he cannot keep his word.

If polygamy is allowed in the kingdom then the knight is out of luck. Whatever statement he makes, the king can keep his word by marrying both daughters to him.

You may think that this problem is just like the recent problem of the week. In fact, it is different. That problem has a solution which allows the king to keep his word only by marrying his younger daughter to the prince. (Can you now figure out what that solution is?) This guarantees a favorable outcome for the prince. So the prince is in a better position in that problem than the knight is in this problem.

In this problem, it is impossible to say what the king will do next. He may decide to punish the irreverent knight, who made him break his promise, by marrying one of his daughters to him anyway. Or maybe he won't want a son-in-law that can outsmart him. The statement gives the knight a chance out of his predicament but does not guarantee the desired outcome. This is because the king is forced to break his promise no matter what. He does not have a chance to keep his word by doing what the knight wants him to do.

Another approach Lorena suggested is that the knight could make such a statement that the king has no way of deciding whether it is true or not. For example, he could say "this kingdom will be attacked by another country in 200 years." This is again either true or false, so is a statement. But the king won't know which for 200 years. That is presumably past his life span.