Math 302 Final Exam Solutions

May 9, 2007

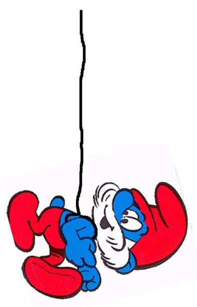
1. (10 pts) Construct the truth table of the logical expression

$$((P \implies \neg Q) \wedge R) \iff ((R \implies P) \wedge (R \implies Q)).$$

Is the expression a tautology, a contradiction, or neither? Why?

P	Q	R	$((P$	$\implies$	$\neg$	$Q)$	$\wedge$	$R)$	$\iff$	$((R$	$\implies$	$P)$	$\wedge$	$(R$	$\implies$	$Q))$
T	T	T	T	F	F	T	F	T	F	T	T	T	T	T	T	T
T	T	F	T	F	F	T	F	F	F	F	T	T	T	F	T	T
T	F	T	T	T	T	F	T	T	F	T	T	T	F	T	F	F
T	F	F	T	T	T	F	F	F	F	F	T	T	T	F	T	F
F	T	T	F	T	F	T	T	T	F	T	F	F	F	T	T	T
F	T	F	F	T	F	T	F	F	F	F	T	F	T	F	T	T
F	F	T	F	T	T	F	T	T	F	T	F	F	F	T	F	F
F	F	F	F	T	T	F	F	F	F	F	T	F	T	F	T	F
				2	1		3		7		4		6		5	

As the truth table shows, this statement is always false, hence it is a contradiction.



2. (15 pts) Consider the following argument.
Every smurf who gathers cranberries lives in a mushroom house. Every smurf who gathers cranberries does not like cats. There is at least one smurf who likes cats or does not live in a mushroom house, and gathers cranberries. Therefore some smurf likes to bungee jump.
If the argument is valid, give a derivation. If it is invalid, explain why it is invalid.

Let

$$\begin{aligned} U &= \text{set of smurfs} \\ P(x) &= x \text{ gathers cranberries} \\ Q(x) &= x \text{ lives in a mushroom house} \\ R(x) &= x \text{ likes cats} \\ S(x) &= x \text{ likes to bungee jump.} \end{aligned}$$

The argument and the derivation are

(1) $(\forall x \in U)\ P(x) \implies Q(x)$

(2) $(\forall x \in U)\ P(x) \implies \neg R(x)$

(3) $(\exists x \in U)\ (R(x) \vee \neg Q(x)) \wedge P(x)$

(4) $(\exists x \in U)\ S(x)$

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(5) $P(a) \implies Q(a)$ — (1), Universal Instantiation

(6) $P(a) \implies \neg R(a)$ — (2), Universal Instantiation

(7) $(R(a) \lor \neg Q(a)) \land P(a)$	(3), Existential Instantiation
(8) $P(a)$	(7), Simplification
(9) $R(a) \lor \neg Q(a)$	(7), Simplification
(10) $\neg R(a)$	(6), (8), Modus Ponens
(11) $\neg Q(a)$	(9), (10) Modus Tollendo Ponens
(12) $Q(a)$	(5), (8), Modus Ponens
(13) $Q(a) \lor S(a)$	(12), Addition
(14) $S(a)$	(13), (11), Modus Tollendo Ponens
(15) $(\exists x \in U)\ S(x)$	(14), Existential Generalization

3. (10 pts each) Prove the following theorems.
   (a) Let a, b be non-negative integers. Show that $a|b$ and $b|a$ if and only if $a = b$. (Hint: if you are tempted to do any division, ask yourself if it is legal first.)

   Suppose $a|b$ and $b|a$. Then $b = ka$ and $a = nb$ for some $k, n \in \mathbb{Z}$. First notice that if $a = 0$ then $b = 0$ and vice versa. Obviously, $a = b$ in this case.
   Otherwise both $a, b > 0$. Hence $k, n > 0$. Substituting $b = ka$ into $a = nb$, we get $a = nb = n(ka) = (nk)a$. We can divide both side by a since $a > 0$. This gives $1 = nk$. Since n and k are positive integers, the only way this can happen is if $n = k = 1$. This shows $a = b$.
   Conversely, suppose $a = b$. Then $a = 1 \cdot b$ and $b = 1 \cdot a$, so $a|b$ and $b|a$.

   (b) If $x, y \in \mathbb{R}$, then $|x + y| \leq |x| + |y|$. (Hint: consider cases depending on the signs of x and y. You can get away with only 3 cases if you are clever, although one of the three may further split into a few subcases.)

   Notice that there are three cases to consider.

   $x, y \geq 0$: Then $x + y \geq 0$. So
   $$|x + y| = x + y = |x| + |y|.$$
   Obviously, then $|x + y| \leq |x| + |y|$.

   $x, y < 0$: Then $x + y < 0$. So
   $$|x + y| = -(x + y) = -x - y = |x| + |y|.$$
   Obviously, then $|x + y| \leq |x| + |y|$.

   $x \geq 0$, $y < 0$: In this case $x + y$ could be positive, negative, or 0. In any case, $|x + y| = \pm(x + y)$ depending on the sign of $x + y$.
   $$y < 0 \implies y < -y \implies x + y < x - y$$
   $$x \geq 0 \implies -x \leq x \implies -x - y \leq x - y$$
   Hence $|x + y| \leq x - y = |x| + |y|$ in either case.

4. (10 pts) Let A, B, C be sets. Prove that if $A \subseteq B$ then $A - C = A \cap (B - C)$. (Hint: remember how to prove that two sets are equal.)

   Let $x \in A - C$. Then $x \in A$ and $x \notin C$. Since $x \in A$ and $A \subseteq B$, $x \in B$. Therefore $x \in B - C$. Hence $x \in A \cap (B - C)$. This shows $A - C \subseteq A \cap (B - C)$.

Let $x \in A \cap (B - C)$. Then $x \in A$ and $x \in B - C$. Since $x \in B - C$¡ $x \in B$ and $x \notin C$. Therefore $x \in A - C$. This shows $A \cap (B - C) \subseteq A - C$.

We can now conclude $A - C = A \cap (B - C)$.

5. (5 pts each) Decide if the following statements are true or false. Justify your answer.

   (a) $\mathcal{P}(\emptyset) \in \mathcal{P}(\mathcal{P}(\emptyset))$.

   True. Any set is a subset of itself, therefore it is an element of its power set. Therefore $\mathcal{P}(\emptyset) \in \mathcal{P}(\mathcal{P}(\emptyset))$.

   (b) $\mathcal{P}(\emptyset) \subseteq \mathcal{P}(\mathcal{P}(\emptyset))$.

   True. First, $\mathcal{P}(\emptyset) = \{\emptyset\}$. Now, $\mathcal{P}(\mathcal{P}(\emptyset)) = \{\emptyset, \{\emptyset\}\}$. Since $\emptyset \in \mathcal{P}(\mathcal{P}(\emptyset))$, therefore $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\}$.

   (c) $S \in \mathcal{P}(S)$ for all sets S.

   True for the same reason as part (a).

   (d) $S \subseteq \mathcal{P}(S)$ for all sets S.

   False. E.g. let $S = \{1\}$. Then $\mathcal{P}(S) = \{\emptyset, \{1\}\}$. Clearly, $1 \notin \mathcal{P}(S)$, so $S = \{1\} \nsubseteq \mathcal{P}(S)$.

   (e) If $S \subseteq T$ then $\mathcal{P}(S) \subseteq \mathcal{P}(T)$.

   True. Let $A \in \mathcal{P}(S)$. Then $A \subseteq S$. Since $S \subseteq T$, $A \subseteq T$. Hence $A \in \mathcal{P}(T)$. This shows $\mathcal{P}(S) \subseteq \mathcal{P}(T)$.

6. (10 pts each) **Extra credit problem.**

   (a) A proverb says "Barking dogs seldom bite." This may or may not be true. Consider the enhanced version of this proverb: "Barking dogs seldom bite unless they bark." Is this true? Why or why not?

   First, let's change "unless" to a more familiar logical operator. In general, "P unless Q" is the same as "P if not Q." So "barking dogs seldom bite unless they bark" is the same as "barking dogs seldom bite if they don't bark." If the premise is true and the dogs don't bark, then the set of barking dogs is empty. Any statement about the empty set is true, including that barking dogs seldom bite. Hence the conclusion is false and the conditional is true.
   If the premise is false, that is the dogs bark, then the conditional is definitely true.
   Therefore "barking dogs seldom bite unless they bark" is a true statement.

   (b) Is there a sequence of 100 consecutive positive integers such that not one of them is a prime number? (Hint: You can solve this problem similarly to the proof that there are infinitely many primes.)

   Sure there is. Let $N = 101! = 101 \cdots 100 \cdots 2 \cdot 1$. Obviously, $2|N$. Also $2|2$, hence $2|(N + 2)$. Since $N > 0$, $N + 2 \neq 2$. So $N + 2$ is a proper multiple of 2. Therefore it is not a prime. The same argument shows that $3|(N + 3)$ and $N + 3 \neq 3$, hence $N + 3$ is not a prime. Continue this way to show that $N + 4, N + 5, \ldots, N + 101$ are not primes. Hence $N + 2, N + 3, \ldots, N + 101$ is a sequence of 100 consecutive integers none of which is prime.