Math 302 Exam 3 Solutions

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1. (10 pts) Let a, b, and c be integers. Suppose that there is some integer d such that $d|a$ and $d|b$, but d does not divide c. Show that the equation $ax + by = c$ has no solution such that x and y are integers.

   Suppose that $d|a$ and $d|b$. We will prove the contrapositive: if there exist $x, y \in \mathbb{Z}$ such that $ax + by = c$ then $d|c$. So suppose such x, y exist. Since $d|a$ and $d|b$, $a = dm$ and $b = dn$ for some $m, n \in \mathbb{Z}$. Hence
   $$c = ax + by = (dm)x + (dn)y = d(mx + ny),$$
   which shows $d|c$.

2. (10 pts) Prove or give a counterexample to each of the following statements.
   (a) For each integer a, there exists an integer b such that $a|b$.

   Let $a \in \mathbb{Z}$. Let $b = a$. Notice that $b = 1a$. Since $1 \in \mathbb{Z}$, $a|b$.

   Alternately, we could just observe that (b) implies (a), and we are about to prove (b).

   (b) There exists an integer b such that for all integers a, we have $a|b$.

   Let $b = 0$. For any $a \in \mathbb{Z}$, $0 = 0a$. Since $0 \in \mathbb{Z}$, $a|0$.

3. (10 pts) Let x and y be real numbers. Define $x \frown y$ and $x \smile y$ to be
   $$x \frown y = \begin{cases} x & \text{if } x \geq y \\ y & \text{if } y \geq x \end{cases}, \qquad \text{and} \qquad x \smile y = \begin{cases} y & \text{if } x \geq y \\ x & \text{if } y \geq x, \end{cases}.$$
   Let a and b be real numbers. Prove that
   $$(a \frown b) - (a \smile b) = |a - b|.$$

   We will split the proof into two cases.
   Case $a \geq b$: In this case $a - b \geq 0$, and
   $$\begin{aligned} a \frown b &= a \\ a \smile b &= b \\ |a - b| &= a - b \end{aligned}$$
   Hence
   $$(a \frown b) - (a \smile b) = a - b = |a - b|.$$
   Case $a < b$: In this case $a - b < 0$, and
   $$\begin{aligned} a \frown b &= b \\ a \smile b &= a \\ |a - b| &= -(a - b) = b - a \end{aligned}$$
   Hence
   $$(a \frown b) - (a \smile b) = b - a = |a - b|.$$

4. (15 pts)
   Let A, B, and C be sets. Prove that
   (a) $A \subseteq A$,

(b) $\emptyset \subseteq A$,
(c) If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

This is Lemma 3.2.2 in your textbook and is proven there.

5. (a) (5 pts) Define what it means for two integers a and b to be relatively prime. Give an example of a pair of relatively prime integers and justify that your numbers are indeed relatively prime.

   Two integers a and b are relatively prime if their only common divisors are ± 1.
   For example, 4 and 15 are relatively prime since the only integers that divide both are ± 1.

   (b) (10 pts) Let $a, b \in \mathbb{Z}$. Prove that if there exist $s, t \in \mathbb{Z}$ such that $sa + tb = 1$, then a and b are relatively prime.

   Suppose there exist $s, t \in \mathbb{Z}$ such that $sa+tb = 1$. Let d be a common divisor of a and b. Then $a = md$ and $b = nd$ for some $m, n \in \mathbb{Z}$. Therefore $1 = sa + tb = s(md) + t(nd) = (sm + tn)d$. Since $m, n, s, t \in \mathbb{Z}$, $sm + tn \in \mathbb{Z}$. It follows that $d|1$. But the only divisors of 1 are ± 1. Hence $d = \pm 1$, which shows that a and b are relatively prime.

   Notice that this proof was quite similar to problem 1. In fact, we could have used the result of problem 1 to give a slightly quicker proof. Here is how.
   We will prove the contrapositive: if a and b are not relatively prime then there exist no $s, t \in \mathbb{Z}$ such that $sa + tb = 1$.
   Suppose a and b have a common divisor $d \neq \pm 1$. Then $d|a$, $d|b$, but obviously $d \nmid 1$ since the only divisors of 1 are ± 1. By problem 1, the equation $sa + tb = 1$ has no solution such that $s, t \in \mathbb{Z}$.

6. (10 pts) **Extra credit problem.** One of your the exercises on your recent homework asked you to show that if A and B are sets such that $A \subseteq B$, then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. Is the converse of this statement true? That is, if A and B are sets such that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, does it follow that $A \subseteq B$? If so, prove it. If you think this is false, give a counterexample.

   This is true, so we will prove it. Let A and B be sets such that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. Since $A \subseteq A$, $A \in \mathcal{P}(A)$. Hence $A \in \mathcal{P}(B)$. But the elements of $\mathcal{P}(B)$ are the subsets of B, so if A is one of those elements then $A \subseteq B$.