Math 303 Exam 2 Solutions

Apr 9, 2008

1. (10 pts) Eight times any triangular number, plus 1, is a square number. Show that this is true for the first four triangular numbers.

$$\begin{aligned}
8T_1 + 1 &= 8\frac{1(1+1)}{2} + 1 = 9 = 3^2 = S_3 \\
8T_2 + 1 &= 8\frac{2(2+1)}{2} + 1 = 25 = 5^2 = S_5 \\
8T_3 + 1 &= 8\frac{3(3+1)}{2} + 1 = 49 = 7^2 = S_7 \\
8T_4 + 1 &= 8\frac{4(4+1)}{2} + 1 = 81 = 9^2 = S_9
\end{aligned}$$

BTW, if you care to see why this happens (the deductive reasoning part), here it is:

$$8T_n + 1 = 8\frac{n(n+1)}{2} + 1 = 4n(n+1) + 1 = 4n^2 + 4n + 1 = (2n+1)^2 = S_{2n+1}.$$

2. (10 pts) Find the smallest integer N greater than 1 such that two different figurate numbers exist for N. What are they?

Let $F_{n,k}$ denote the k-th n-gonal number, where $n \geq 3$ and $k \geq 1$. E.g. $F_{3,2}$ stands for the second triangular number. First, we will make a few observations about these numbers:

$$\begin{aligned}
1 = F_{3,1} = F_{4,1} = F_{5,1} = F_{6,1} = \cdots& \\
F_{3,k} < F_{4,k} < F_{5,k} < F_{6,k} < \cdots& \qquad \text{for any } k > 1 \\
1 = F_{n,1} < F_{n,2} < F_{n,3} < F_{n,4} < \cdots& \qquad \text{for any } n \geq 3
\end{aligned}$$

All these observations come from thinking about the meaning of these numbers.

By the first inequality, $F_{n,k} \leq F_{n',k}$ for $3 \leq n \leq n'$ with equality if and only if $n = n'$. By the second inequality, $F_{n',k} \leq F_{n',k'}$ for $1 < k \leq k'$ with equality if and only if $k = k'$. Combining these, we get

$$F_{n,k} \leq F_{n,k'} \leq F_{n',k'} \text{ if } 3 \leq n \leq n' \text{ and } 1 < k \leq k'$$

with equality if and only if $n = n'$ and $k = k'$.

The smallest values of interest to us are $n = 3$ and $k = 2$. Then $F_{3,2} = 3$. By the last inequality, any other figurate number that is greater than 1 will be strictly larger than 3. So 3 is out. What about 4 and 5? Indeed, $F_{4,2} = 4$ and $F_{5,2} = 5$. Could 4 or 5 show up again among the triangular numbers? No, because $F_{3,2} = 3 \neq 4, 5$ and $F_{3,3} = 6$, so $F_{3,k} \leq 6$ whenever $k \leq 3$. Now $F_{n,k} > F_{4,2} = 4$ whenever $n > 4$ and $k \geq 2$ or $n \geq 4$ and $k > 2$. So 4 cannot appear again as a figurate number among the square, pentagonal, hexagonal, etc numbers. Similarly, $F_{n,k} > F_{5,2} = 5$ whenever $n > 5$ and $k \geq 2$ or $n \geq 5$ and $k > 2$. So 5 cannot appear again as a figurate number among the pentagonal, hexagonal, etc numbers. Neither does 5 appear among the square numbers as $F_{4,3} = 9$ and so $F_{4,k} > 9$ when $k > 3$.

This leaves $6 = F_{3,3} = F_{6,2}$ as the smallest number (> 1) which shows up twice among the figurate numbers. It is both a triangular and a hexagonal number.

Note: many of you found that 6 was both a triangular and a hexagonal number, but neglected to justify why it was the smallest possible number to play such a dual role.

3. (10 pts) Choose any three-digit number with all different digits. Now reverse the digits. Of your original three-digit number and the reversed one, subtract the smaller from the larger. Record your result. Repeat this with as many three-digit numbers as you need to see the pattern.
   (a) What pattern do you observe?

   The pattern is that you always get multiples of 99 this way. Alternately, you may have observed that the middle digit is always 9 and the sum of the first and third digits is also 9. Or that the middle digit is 9 and the sum of all three digits is 18. These are all equivalent observations.

   (b) Write an explanation for this pattern. Why do you always get the kind of number you do?

   I will explain why the result is a multiple of 99. If you observed a different pattern in part (a), then you would have to explain that.
   Let the digits of the starting number be x, y, z. That is we are talking about the number $100x + 10y + z$. When we reverse the digits, we get $100z + 10y + x$. Since I don't know which number is bigger, I will take the absolute value of the difference:

$$\begin{aligned} |(100x + 10y + z) - (100z + 10y + x)| &= |100x - x + z - 100z| \\ &= |99(x - z)| = 99|x - z|. \end{aligned}$$

   which is indeed a multiple of 99. Incidentally, I also showed that you can get any multiple of 99 between 99 and $9 \cdot 99 = 991$ by choosing x and z appropriately.

   Note that I didn't even to know that the digits were all different. This is only needed to make sure that there is a bigger and a smaller number. If $x = z$, you just get 0 as the result, which is still a multiple of 99. The middle digit could be anything you want.

4. (15 pts)
   (a) When and where did Euclid live? What was his nationality? (A century and a country-size region of the world will suffice.)

   He lived in Alexandria (Egypt) around 300 BC. He was Greek.

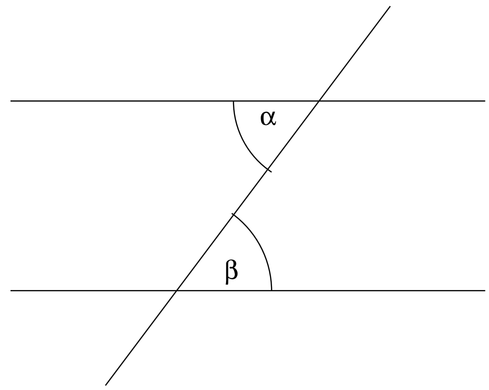
   (b) State Euclid's five postulates. You can use Euclid's original wording or modern language.

   See pp. 34–35 in Journey through Genius.

   (c) Euclid was first to develop an axiomatic treatment of geometry. State two major advantages of such a system.

   Such a system avoids the danger of circular reasoning, by stating explicitly what axioms are accepted without proof and logically deriving every other result from these.
   It also makes it easy to follow what changes if an axiom is changed or deleted. This is because one can trace it back which axioms a theorem (proposition) depends on.

5. (15 pts) Euclid's proposition I.29 says "a straight line falling on parallel straight lines makes the alternate angles equal to each other." (The alternate angles referred to are α and β in the figure below.) Prove this proposition. You may assume that the following result is already known: The sum of two adjacent angles on a straight line is two right angles.



This is Proposition I.27 of the Elements. You can find the proof in Journey through Genius.

6. (15 pts) **Extra credit problem.** We pointed out in class that Euclid's axioms allow the use of a collapsible compass. That is one that can draw circles with any center and any radius, but once lifted up from the paper, it falls shut. Therefore it cannot be used to copy lengths. We also said that Euclid proved in proposition I.3 in his Elements that it is possible to copy lengths. That is the compass can stay open when lifted from the paper. State and prove such a proposition.

This is not easy to do. As I mentioned in class–and this much is in the book–this was Proposition I.3 in Euclid's Elements. So at least we can guess that we do not need to prove a long list of other results before we can attack this one.

The statement will have to say something about copying a given line segment onto another line starting at some given point on that line. Actually, we may need to specify which side of that point we want the copied sequence to be on. Here is one possible statement:

Proposition: Given two line segments AB and CD such that AB is longer than CD, it is possible to construct a point E on AB such that AE is the same length as CD.

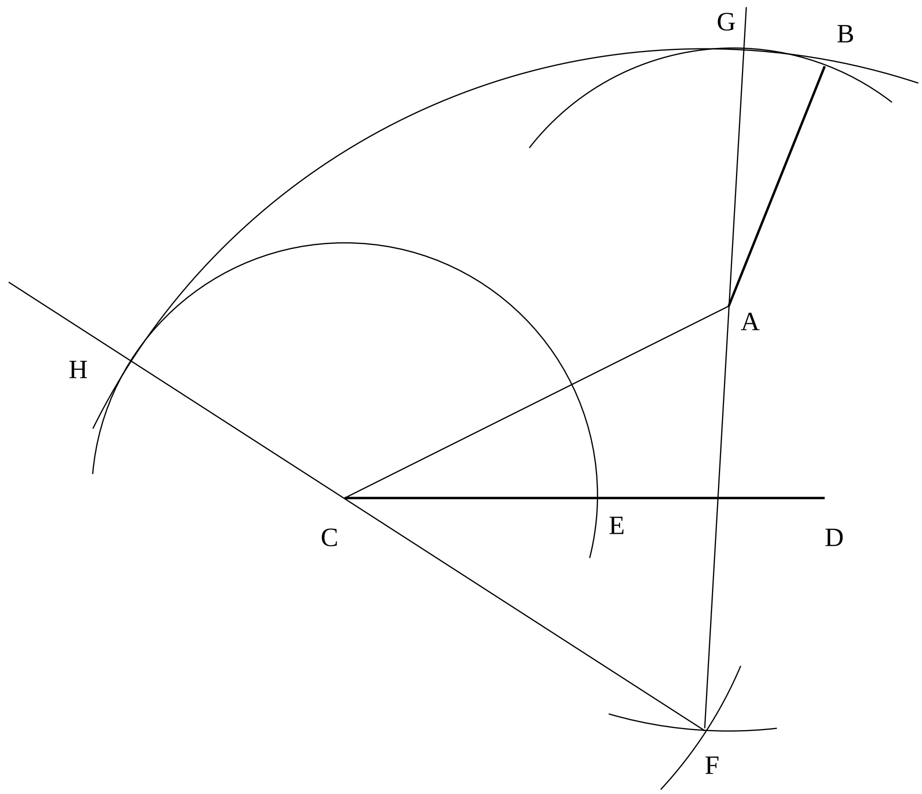
Proof: Here is how Euclid would have proven this. Refer to the figure below.

First, connect A and C. This can be done by Postulate 1. Now construct the point F so that ACF is an equilateral triangle. We did such a thing in class and all we needed for it was Postulate 3, which allows us to draw a circle centered at C with radius AC and a circle centered at A with radius AC. Using Postulates 1 and 2, connect F and C and extend this line segment beyond C. Similarly, connect F and A and extend this line segment beyond A. Draw a circle centered at A with radius AB. Let G be the point where this circle intersects the ray FA. (A ray is a line segment extended indefinitely in one direction.) Draw a circle centered at F with radius FG. Let H be the point where this circle intersects the ray FC. Finally, draw a circle centered at C with radius CH. Let E be the point where this intersects CD.

We claim that $\overline{CE} = \overline{AB}$. First, $\overline{AG} = \overline{AB}$ by the definition of circle. Next, $\overline{FH} = \overline{FG}$ for the same reason. Remember that $\overline{FC} = \overline{FA}$ because ACF was constructed to be an equilateral triangle. Now $\overline{CH} = \overline{FH} - \overline{FC} = \overline{FG} - \overline{FA} = \overline{AG}$ by Common Notion 3. Finally, $\overline{CE} = \overline{CH}$ by the definition of circle. Now a few invocations of Common Notion 1 gives

$$\overline{CE} = \overline{CH} = \overline{AG} = \overline{AB}$$

as desired.



Note: What we just established is that you can transfer lengths from one line segment to another. In practice, we do this with a compass which we keep open as we transfer the length. You could of course just as well do it with a piece of string. The challenge is not to design a suitable physical device to do this, but to prove that the use of such a device is consistent with the axioms of Euclidean geometry.