Math 303 Final Exam Solutions

May 12, 2008

1. (10 pts) Give conditions under which the following equation would be true

$$a + (b - c) = (a + b) - (a + c).$$

This is just a matter of solving the equation.

$$\begin{aligned} a + (b - c) &= (a + b) - (a + c) \\ a + b - c &= a + b - a - c \\ a + b - c &= b - c \\ a &= 0 \end{aligned}$$

Since all the steps we took (rearranging and cancellation) were reversible, $a = 0$ is both necessary and sufficient. So the equation is true if and only if $a = 0$ without any restriction on b and c.

2. (15 pts) Consider the following table:

J	m	n	p
m	n	p	n
n	p	m	n
p	n	n	m

(a) Is J an operation on the set $S = \{m, n, p\}$ (i.e. is S closed under J)? Why?

Yes, it is. Every entry in the body of the table is in S, so S is closed under J. That is, J is an operation on S.

(b) Is J commutative? Why?

The table is symmetric with respect to the diagonal from the upper left to the lower right corner. This shows that $xJy = yJx$ for any x, and y in S. Hence J is commutative.

(c) Does J have an identity? Why?

No, it does not. If there were an identity then the row and column corresponding to that element would have to be copies of the header row and column of the table. None of the rows (or columns) is a copy of the header row (or column), hence there is no identity.

(d) Does each element have an inverse? Why?

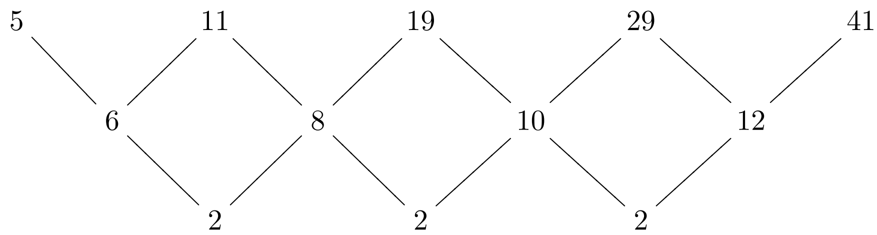
Since there is no identity, no element can have an inverse.

3. (15 pts) Suppose the expression $n^2 + 3n + 1$ determines the n-th term in a sequence. That is, to find the first term, let $n = 1$; to find the second term, let $n = 2$, and so on.

(a) Find the first four terms of the sequence.

n	1	2	3	4
$n^2 + 3n + 1$	5	11	19	29

(b) Use the method of successive differences to predict the 5th term of the sequence.



(c) Find the 5th term of the sequence by letting $n = 5$ in the expression $n^2 + 3n + 1$. Does your result agree with the one you found in part (b)?

It is $5^2 + 3 \cdot 5 + 1 = 41$. Yes, it does, as it should.

4. (10 pts)
(a) What does it mean for line segments $\overline{AB}$ and $\overline{CD}$ to be commensurable?

Two line segments $\overline{AB}$ and $\overline{CD}$ are commensurable if there exists a line segment $\overline{EF}$ which fits into each $\overline{AB}$ and $\overline{CD}$ an integer number of times.

(b) The Pythagoreans (and other Greek mathematicians prior to them) assumed that any two line segments were commensurable. As we pointed out in class, this is equivalent to assuming that every number is rational. Show that this is so: two line segments are commensurable if and only if the ratio of their lengths is rational.

Let us do the "only if" part first. Suppose $\overline{AB}$ and $\overline{CD}$ are commensurable. Then there exists a line segment $\overline{EF}$ and two integers m, n such that $\overline{AB} = m\overline{EF}$ and $\overline{CD} = n\overline{EF}$. So
$$\frac{\overline{AB}}{\overline{CD}} = \frac{m\overline{EF}}{n\overline{EF}} = \frac{m}{n},$$
which is clearly a rational number.
Here is the "if" part. Suppose that $\overline{AB}/\overline{CD}$ is rational. That is there exist integers m, n such that $n \neq 0$ and $\overline{AB}/\overline{CD} = m/n$. Since the ratio of the lengths of two line segments must be a positive number, we may as well assume both m and n are positive. (If they are both negative, just switch their signs.) Now take $1/m$ part of $\overline{AB}$ and call it $\overline{EF}$. It follows that
$$\frac{\overline{AB}}{\overline{CD}} = \frac{m}{n} \implies \frac{\overline{AB}}{m} = \frac{\overline{CD}}{n} = \overline{EF}.$$
This shows that $\overline{EF}$ fits into $\overline{AB}$ m times and into $\overline{CD}$ n times.

5. (12 pts)
(a) Why is Euclid's Elements seen as a significant step in the history of mathematics?

It was the first systematic treatment of mathematics which proceeded from clearly stated axioms (postulates and common notions) to theorems (propositions) so that the proofs used only axioms and previously proved results. This ensured that the proofs were not circular and made it clear how the theory depended on each particular axiom.

(b) State Euclid's 5th postulate.

If a line intersecting two other lines is such that the angles on one side sum to less than two right angles, then the two lines intersect on the same side that the angles are on.

(c) Euclid's geometry existed as the only logically consistent geometry for a long time after the publication of the elements. Eventually, hyperbolic geometry was invented as the first non-Euclidean geometry by replacing the 5th postulate with a statement about the sum of the internal angles of the triangle. About when did this happen? Which three mathematicians do we credit for this work?

It happened around 1830 and was done by Carl Friedrich Gauss, János Bólyai, and Nikolai Ivanovich Lobachevski.

(d) What statement about the sum of the angles of a triangle replaces the 5th postulate in hyperbolic geometry?

The sum of the internal angles of a triangle is less than 180°.

6. (10 pts) One of Euclid's propositions in the 7th book of the Elements says "Any composite number is measured by some prime number." Prove this.

First, note that "measured by" is Euclid's way of saying "divisible by."

Let x be a composite number. Then x must have a proper divisor $1 < x_1 < x$. If x_1 is prime, we are done. Otherwise, x_1 has a proper divisor $1 < x_2 < x_1$. Since x_2 divides x_1, it must also divide x. Continue this way as long as you can to obtain a decreasing sequence

$$x > x_1 > x_2 > \ldots > 1$$

of divisors. This sequence must be finite because there are only finitely many integers between 1 and x. Therefore there is a last element x_n. This must be prime, otherwise the sequence could be continued. This x_n is a prime divisor of x_{n-1}, hence of x_{n-2}, hence of x_{n-3}, etc. Finally, it is a prime divisor of x.

7. (8 pts)

(a) Finding the first formula for solving the general cubic equation was the work of three mathematicians. Who were they? When and where did they live? (A century and a country-size region of the world will suffice.)

Scipione del Ferro, Niccolò Fontana (Tartaglia), and Gerolamo Cardano. They lived around 1500 and were all Italian.

(b) What is the depressed cubic equation?

$$x^3 + ax + b = 0.$$

Alternately, you could write $x^3 + mx = n$ or even $ax^3 + bx + c = 0$ would acceptable. The important thing is that it is a cubic equation which does not have a quadratic term.

8. (10 pts) **Extra credit problem.** Prove that the number $2^n(2^{n+1} - 1)$ is perfect if $2^{n+1} - 1$ is prime.

Suppose $2^{n+1} - 1$ is prime and for now, denote it by p. We need to show that $2^n p$ is perfect. The divisors of such a number are

$$1, 2, 4, \ldots, 2^n, p, 2p, 4p, \ldots 2^n p.$$

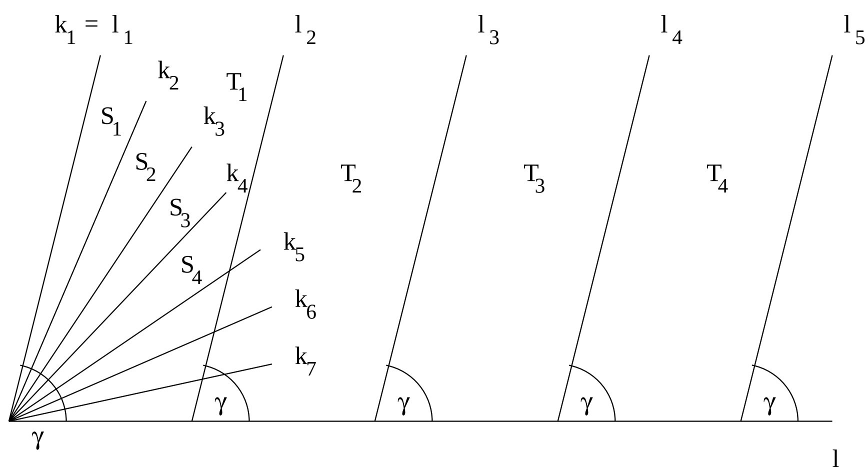
So the sum of all divisors is

$$\sum_{i=0}^{n} 2^i + \sum_{i=0}^{n} 2^i p = (1+p)\sum_{i=0}^{n} 2^i = (1+p)(2^{n+1}-1) = 2^{n+1}(2^{n+1}-1),$$

which is twice $2^n(2^{n+1}-1)$. So if we do not include $2^n(2^{n+1}-1)$ in the sum, we get exactly $2^n(2^{n+1}-1)$. Hence this number is perfect.

9. (10 pts) **Extra credit problem.** Mathematicians tried for many centuries to prove that Euclid's 5th postulate followed from the other postulates and common notions. Here is one attempt at proving this.

   First consider the figure below, in which the rays $l_1, l_2, \ldots$ all have the same angle γ with the ray l, and the rays $k_1, k_2, \ldots, k_n$ divide γ into n equal parts.

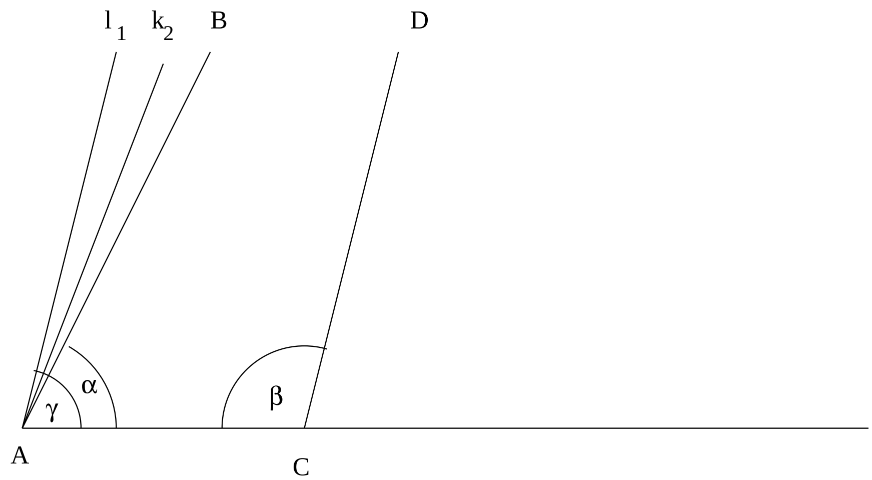


   Let S_i be the sector enclosed by k_i and k_{i+1}. Then the areas of all the S_i are equal and infinite. Notice that when you combine them, you get the area of the sector enclosed by k_1 and l.

   Now, let T_i be the strip enclosed by the rays l_i, k, l_{i+1}. The areas of the T_i are equal and infinite. When you combine them, you get the area of the sector enclosed by l_1 and l.

   So the combined areas of the S_i is equal to the combined areas of the T_i. But there are only finitely many S_i and infinitely many T_i, so the area of S_i must be bigger than the area of T_i, otherwise we would need infinitely many of the sectors too to make up the sector enclosed by k_1 and l. In particular, S_1 is bigger than T_1, so it cannot fit entirely inside T_1. Therefore k_2 must intersect l_2. This is true independently of how wide the strips are and how many parts γ is divided into.

   Now, let the lines AB, CD, and AC be as in the figure below with $\alpha + \beta < 180°$. Let $\gamma = 180° - \beta$ and let l_1 be a ray starting at A such that its angle with the ray AC is γ.



Since $180° - \alpha - \beta > 0°$, we can choose a positive integer n such that $\gamma/n < 180° - \alpha - \beta$. Now divide γ into n equal parts and use the previous result. It says k_2 must intersect $l_2 = CD$. Therefore AB must also intersect CD, which is what we wanted to prove.

We know that the 5th postulate does not have to hold. So the above "proof" must have a mistake. Find the mistake in it.

The problem is with the comparison of infinite areas. It sounds intuitive that the infinite area of S_i must be bigger than the infinite area of the T_i because we only need finitely many of the sectors to cover the same area as infinitely many of the strips, but this is not based on any rigorous mathematical principle. It is like saying that there are more rational numbers than integers because the integers are only a small part of the set of the rational numbers. In fact, as you probably know, the set of integers and the set of rationals are the same size.