1.2 An Application of Inductive Reasoning: Number Patterns

In the previous section we introduced inductive reasoning, and we showed how it can be applied in predicting "what comes next" in a list of numbers or equations. In this section we will continue our investigation of number patterns.

An ordered list of numbers, such as

$$3, 9, 15, 21, 27, \ldots,$$

is called a *sequence*. A **number sequence** is a list of numbers having a first number, a second number, a third number, and so on, called the **terms** of the sequence. To indicate that the terms of a sequence continue past the last term written, we use three dots (ellipsis points). The sequences in Examples 2(a) and 2(c) in the previous section are called *arithmetic* and *geometric sequences,* respectively. An arithmetic sequence has a common *difference* between successive terms, while a geometric sequence has a common *ratio* between successive terms. The Fibonacci sequence in Example 2(b) is covered in a later chapter.

Successive Differences

The sequences seen in the previous section were usually simple enough for us to make an obvious conjecture about the next term. However, some sequences may provide more difficulty in making such a conjecture, and often the **method of successive differences** may be applied to determine the next term if it is not obvious at first glance. For example, consider the sequence

$$2, 6, 22, 56, 114, \ldots.$$

Since the next term is not obvious, subtract the first term from the second term, the second from the third, the third from the fourth, and so on.

2		6		22		56		114
	$6 - 2 = 4$		$22 - 6 = 16$		$56 - 22 = 34$		$114 - 56 = 58$	

Now repeat the process with the sequence 4, 16, 34, 58 and continue repeating until the difference is a constant value, as shown in line (4):

2		6		22		56		114	**(1)**
	4		16		34		58		**(2)**
		12		18		24			**(3)**
			6		6				**(4)**

Once a line of constant values is obtained, simply work "backward" by adding until the desired term of the given sequence is obtained. Thus, for this pattern to continue, another 6 should appear in line (4), meaning that the next term in line (3) would have to be $24 + 6 = 30$. The next term in line (2) would be $58 + 30 = 88$. Finally, the next term in the given sequence would be $114 + 88 = 202$. The final scheme of numbers is shown below.

2		6		22		56		114		202		**(1)**
	4		16		34		58		88			**(2)**
		12		18		24		30				**(3)**
			6		6		6					**(4)**

YOU TRY IT

EXAMPLE 1 Use the method of successive differences to determine the next number in each sequence.

(a) $14, 22, 32, 44, \ldots$

Using the scheme described above, obtain the following:

14		22		32		44		58
	8		10		12		14	
		2		2		2		

Once the row of 2s was obtained and extended, we were able to get $12 + 2 = 14$, and $44 + 14 = 58$, as shown above. The next number in the sequence is 58.

(b) $5, 15, 37, 77, 141, \ldots$

Proceeding as before, obtain the following diagram.

5		15		37		77		141		235
	10		22		40		64		94	
		12		18		24		30		
			6		6		6			

The next number in the sequence is 235. ■

The method of successive differences will not always work. For example, try it on the Fibonacci sequence in Example 2(b) of Section 1.1 and see what happens!

Number Patterns One of the most amazing aspects of mathematics is its seemingly endless variety of number patterns. Observe the following pattern:

$$1 = 1^2$$
$$1 + 3 = 2^2$$
$$1 + 3 + 5 = 3^2$$
$$1 + 3 + 5 + 7 = 4^2$$
$$1 + 3 + 5 + 7 + 9 = 5^2.$$

In each case, the left side of the equation is the indicated sum of the consecutive odd counting numbers beginning with 1, and the right side is the square of the number of terms on the left side. You should verify this in each case. Inductive reasoning would suggest that the next line in this pattern is

$$1 + 3 + 5 + 7 + 9 + 11 = 6^2.$$

Evaluating each side shows that each side simplifies to 36.

Can we conclude from these observations that this pattern will continue indefinitely? The answer is no, because observation of a finite number of examples does not guarantee that the pattern will continue. However, mathematicians have proved that this pattern does indeed continue indefinitely, using a method of proof called *mathematical induction.* (See any standard college algebra text.)

Any even counting number may be written in the form $2k$, where k is a counting number. It follows that the kth odd counting number is written $2k - 1$. For example, the third odd counting number, 5, can be written $2(3) - 1$. Using these ideas, we can write the result obtained above as follows.

Sum of the First *n* Odd Counting Numbers

If n is any counting number,

$$\mathbf{1 + 3 + 5 + \cdots + (2n - 1) = n^2.}$$

EXAMPLE 2 In each of the following, several equations are given illustrating a suspected number pattern. Determine what the next equation would be, and verify that it is indeed a true statement.

(a)

$$1^2 = 1^3$$
$$(1 + 2)^2 = 1^3 + 2^3$$
$$(1 + 2 + 3)^2 = 1^3 + 2^3 + 3^3$$
$$(1 + 2 + 3 + 4)^2 = 1^3 + 2^3 + 3^3 + 4^3$$

The left side of each equation is the square of the sum of the first n counting numbers, while the right side is the sum of their cubes. The next equation in the pattern would be

$$(1 + 2 + 3 + 4 + 5)^2 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3.$$

Each side of the above equation simplifies to 225, so the pattern is true for this equation.

(b)

$$\begin{aligned} 1 &= 1^3 \\ 3 + 5 &= 2^3 \\ 7 + 9 + 11 &= 3^3 \\ 13 + 15 + 17 + 19 &= 4^3 \end{aligned}$$

The left sides of the equations contain the sum of odd counting numbers, starting with the first (1) in the first equation, the second and third (3 and 5) in the second equation, the fourth, fifth, and sixth (7, 9, and 11) in the third equation, and so on. The right side contains the cube (third power) of the number of terms on the left side in each case. Following this pattern, the next equation would be

$$21 + 23 + 25 + 27 + 29 = 5^3,$$

which can be verified by computation.

(c)

$$\begin{aligned} 1 &= \frac{1 \cdot 2}{2} \\ 1 + 2 &= \frac{2 \cdot 3}{2} \\ 1 + 2 + 3 &= \frac{3 \cdot 4}{2} \\ 1 + 2 + 3 + 4 &= \frac{4 \cdot 5}{2} \end{aligned}$$

The left side of each equation gives the indicated sum of the first n counting numbers, and the right side is always of the form

$$\frac{n(n + 1)}{2}.$$

For the pattern to continue, the next equation would be

$$1 + 2 + 3 + 4 + 5 = \frac{5 \cdot 6}{2}.$$

Since each side simplifies to 15, the pattern is true for this equation.

The patterns established in Examples 2(a) and 2(c) can be written as follows.

Two Special Sum Formulas

For any counting number n,

$$(1 + 2 + 3 + \cdots + n)^2 = 1^3 + 2^3 + 3^3 + \cdots + n^3$$

and

$$1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}.$$

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Pythagoras The Greek mathematician Pythagoras lived during the sixth century B.C. He and his fellow mathematicians formed the Pythagorean brotherhood, devoted to the study of mathematics and music. The Pythagoreans investigated the figurate numbers introduced in this section. They also discovered that musical tones are related to the lengths of stretched strings by ratios of counting numbers. You can test this on a cello. Stop any string midway, so that the ratio of the whole string to the part is 2/1. If you pluck the free half of the string, you get the octave above the fundamental tone of the whole string. The ratio 3/2 gives you the fifth above the octave, and so on. The string ratio discovery was one of the first concepts of mathematical physics.

The second formula given is a generalization of the method first explained preceding Exercise 43 in the previous section, relating the story of young Carl Gauss. We can provide a general, deductive argument showing how this equation is obtained. Suppose that we let S represent the sum $1 + 2 + 3 + \cdots + n$. This sum can also be written as $S = n + (n - 1) + (n - 2) + \cdots + 1$. Now write these two equations as follows.

$$\begin{array}{rcl} S &=& 1 \quad + 2 \quad + 3 \quad + \cdots + n \\ S &=& n \quad + (n-1) + (n-2) + \cdots + 1 \\ \hline 2S &=& (n+1) + (n+1) + (n+1) + \cdots + (n+1) \end{array}$$

Add the corresponding sides.

Since the right side of the equation has n terms, each of them being $(n + 1)$, we can write it as n times $(n + 1)$.

$$2S = n(n + 1)$$

$$S = \frac{n(n + 1)}{2}$$

Divide both sides by 2.

Now that this formula has been verified in a general manner, we can apply deductive reasoning to find the sum of the first n counting numbers for any given value of n. (See Exercises 21–24.)

Figurate Numbers The Greek mathematician Pythagoras (c. 540 B.C.) was the founder of the Pythagorean brotherhood. This group studied, among other things, numbers of geometric arrangements of points, such as *triangular numbers*, *square numbers*, and *pentagonal numbers*. Figure 5 illustrates the first few of each of these types of numbers.

The *figurate numbers* possess numerous interesting patterns. Every square number greater than 1 is the sum of two consecutive triangular numbers. (For example, $9 = 3 + 6$ and $25 = 10 + 15$.) Every pentagonal number can be represented as the sum of a square number and a triangular number. (For example, $5 = 4 + 1$ and $12 = 9 + 3$.)

In the expression T_n, n is called a subscript. T_n is read "T sub n," and it represents the triangular number in the nth position in the sequence. For example,

$$T_1 = 1, \quad T_2 = 3, \quad T_3 = 6, \quad \text{and} \quad T_4 = 10.$$

S_n and P_n represent the nth square and pentagonal numbers respectively.

Formulas for Triangular, Square, and Pentagonal Numbers

For any natural number n,

the nth triangular number is given by $\mathbf{T_n = \dfrac{n(n + 1)}{2}}$,

the nth square number is given by $\mathbf{S_n = n^2}$, and

the nth pentagonal number is given by $\mathbf{P_n = \dfrac{n(3n - 1)}{2}}$.

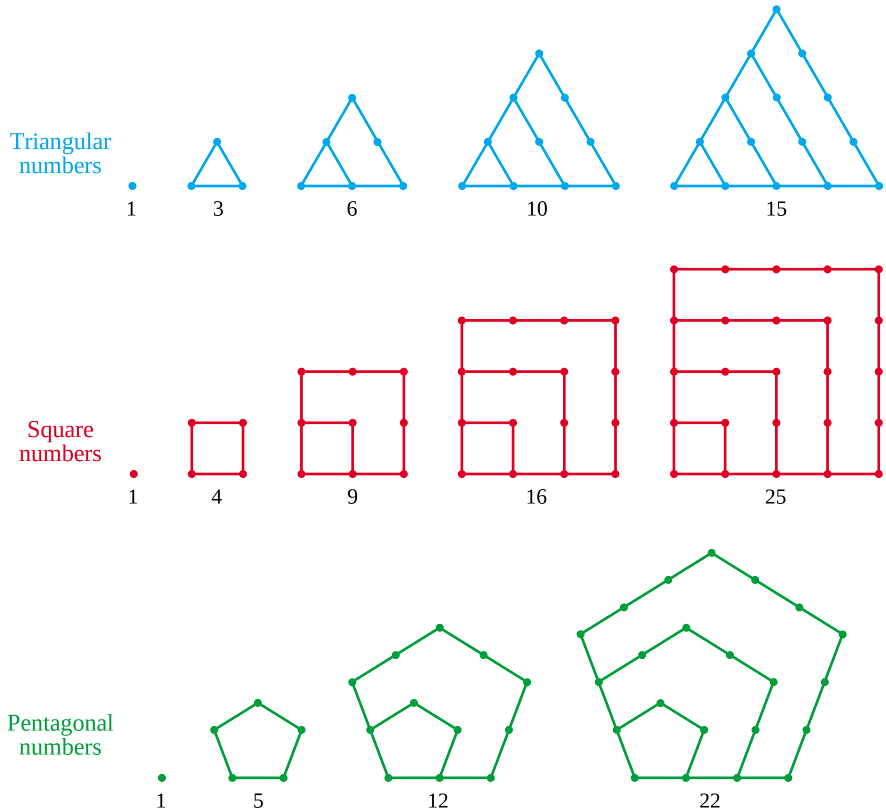

FIGURE 5

EXAMPLE 3 Use the formulas to find each of the following.

(a) the seventh triangular number

$$T_7 = \frac{7(7+1)}{2} = \frac{7(8)}{2} = \frac{56}{2} = 28$$

(b) the twelfth square number

$$S_{12} = 12^2 = 144$$

(c) the sixth pentagonal number

$$P_6 = \frac{6[3(6)-1]}{2} = \frac{6(18-1)}{2} = \frac{6(17)}{2} = 51$$

EXAMPLE 4 Show that the sixth pentagonal number is equal to 3 times the fifth triangular number, plus 6.

From Example 3(c), $P_6 = 51$. The fifth triangular number is 15. According to the problem,

$$51 = 3(15) + 6 = 45 + 6 = 51.$$

The general relationship examined in Example 4 can be written as follows:

$$P_n = 3 \cdot T_{n-1} + n \qquad (n \geq 2).$$

Other such relationships among figurate numbers are examined in the exercises of this section.

The method of successive differences, introduced at the beginning of this section, can be used to predict the next figurate number in a sequence of figurate numbers.

EXAMPLE 5 The first five pentagonal numbers are

1, 5, 12, 22, 35.

Use the method of successive differences to predict the sixth pentagonal number.

1	5	12	22	35	**51**
	4	7	10	13	**16**
		3	3	3	**3**

After the second line of successive differences, we can work backward to find that the sixth pentagonal number is 51, which was also found in Example 3(c).

FOR FURTHER THOUGHT

Take any three-digit number whose digits are not all the same. Arrange the digits in decreasing order, and then arrange them in increasing order. Now subtract. Repeat the process, using a 0 if necessary in the event that the difference consists of only two digits. For example, suppose that we choose the digits 1, 4, and 8.

$$\begin{array}{r} 841 \\ -148 \\ \hline 693 \end{array} \qquad \begin{array}{r} 963 \\ -369 \\ \hline 594 \end{array} \qquad \begin{array}{r} 954 \\ -459 \\ \hline 495 \end{array}$$

Notice that we have obtained the number 495, and the process will lead to 495 again. The number 495 is called a Kaprekar number, and it will eventually always be generated if this process is followed for such a three-digit number.

For Group Discussion

1. Have each student in a group of students apply the process of Kaprekar to a different two-digit number, in which the digits are not the same. (Interpret 9 as 09 if necessary.) Compare the results. What seems to be true?
2. Repeat the process for four digits, with each student in the group comparing results after several steps. What conjecture can the group make for this situation?