1.2 EXERCISES

Use the method of successive differences to determine the next number in each sequence.

- **1.** 1, 4, 11, 22, 37, 56, ...
- **3.** 6, 20, 50, 102, 182, 296, ...
- **5.** 0, 12, 72, 240, 600, 1260, 2352, ...
- **7.** 5, 34, 243, 1022, 3121, 7770, 16799,...

- **2.** 3, 14, 31, 54, 83, 118, ...
- **4.** 1, 11, 35, 79, 149, 251,...
- **6.** 2, 57, 220, 575, 1230, 2317,...
- **8.** 3, 19, 165, 771, 2503, 6483, 14409,...

9. Refer to Figures 2 and 3 in the previous section. The method of successive differences can be applied to the sequence of interior regions,

to find the number of regions determined by seven points on the circle. What is the next term in this sequence? How many regions would be determined by eight points? Verify this using the formula given at the end of that section.

- 10. Suppose that the expression $n^2 + 3n + 1$ determines the *n*th term in a sequence. That is, to find the first term, let n = 1; to find the second term, let n = 2, and so on.
 - (a) Find the first four terms of the sequence.
 - (**b**) Use the method of successive differences to predict the fifth term of the sequence.
 - (c) Find the fifth term of the sequence by letting n = 5 in the expression $n^2 + 3n + 1$. Does your result agree with the one you found in part (b)?

In each of the following, several equations are given illustrating a suspected number pattern. Determine what the next equation would be, and verify that it is indeed a true statement.

11. $(1 \times 9) - 1 = 8$ 12. $(1 \times 8) + 1 = 9$ $(21 \times 9) - 1 = 188$ $(12 \times 8) + 2 = 98$ $(321 \times 9) - 1 = 2888$ $(123 \times 8) + 3 = 987$ **13.** 999,999 \times 2 = 1,999,998 **14.** $101 \times 101 = 10,201$ $999,999 \times 3 = 2,999,997$ $10,101 \times 10,101 = 102,030,201$ 15. $3^2 - 1^2 = 2^3$ 16. $1 = 1^2$ $6^2 - 3^2 = 3^3$ $1 + 2 + 1 = 2^2$ $10^2 - 6^2 = 4^3$ $1 + 2 + 3 + 2 + 1 = 3^2$ $15^2 - 10^2 = 5^3$ $1 + 2 + 3 + 4 + 3 + 2 + 1 = 4^{2}$ **17.** $2^2 - 1^2 = 2 + 1$ **18.** $1^2 + 1 = 2^2 - 2$ $3^{2} - 2^{2} = 3 + 2$ $4^{2} - 3^{2} = 4 + 3$ $2^2 + 2 = 3^2 - 3$ $3^2 + 3 = 4^2 - 4$ **19.** $1 = 1 \times 1$ **20.** 1 + 2 = 3 $1 + 5 = 2 \times 3$ 4 + 5 + 6 = 7 + 8 $1 + 5 + 9 = 3 \times 5$ 9 + 10 + 11 + 12 = 13 + 14 + 15

Use the formula $S = \frac{n(n+1)}{2}$ derived in this section to find each of the following sums.

21. $1 + 2 + 3 + \dots + 300$ **22.** $1 + 2 + 3 + \dots + 500$ **23.** $1 + 2 + 3 + \dots + 675$ **24.** $1 + 2 + 3 + \dots + 825$

Use the formula $S = n^2$ discussed in this section to find each of the following sums. (Hint: To find n, add 1 to the last term and divide by 2.)

- **25.** 1 + 3 + 5 + · · · + 101
- **27.** 1 + 3 + 5 + · · · + 999
- **29.** Use the formula for finding the sum

$$1+2+3+\cdots+n$$

to discover a formula for finding the sum

$$2 + 4 + 6 + \cdots + 2n$$

- **26.** $1 + 3 + 5 + \dots + 49$
- **28.** $1 + 3 + 5 + \cdots + 301$
- **30.** State in your own words the following formula discussed in this section:

$$(1 + 2 + 3 + \dots + n)^2 = 1^3 + 2^3 + 3^3 + \dots + n^3$$

31. Explain how the following diagram geometrically illustrates the formula $1 + 3 + 5 + 7 + 9 = 5^2$.



32. Explain how the following diagram geometrically illustrates the formula $1 + 2 + 3 + 4 = \frac{4 \times 5}{2}$.



33. Use patterns to complete the table below.

Figurate Number	1st	2nd	3rd	4th	5th	6th	7th	8th
Triangular	1	3	6	10	15	21		
Square	1	4	9	16	25			
Pentagonal	1	5	12	22				
Hexagonal	1	6	15					
Heptagonal	1	7						
Octagonal	1							

34. The first five triangular, square, and pentagonal numbers may be obtained using sums of terms of sequences, as shown below.

Triangular	Square	Pentagonal
1 = 1	1 = 1	1 = 1
3 = 1 + 2	4 = 1 + 3	5 = 1 + 4
6 = 1 + 2 + 3	9 = 1 + 3 + 5	12 = 1 + 4 + 7
10 = 1 + 2 + 3 + 4	16 = 1 + 3 + 5 + 7	22 = 1 + 4 + 7 + 10
15 = 1 + 2 + 3 + 4 + 5	25 = 1 + 3 + 5 + 7 + 9	35 = 1 + 4 + 7 + 10 + 13

Notice the successive differences of the added terms on the right sides of the equations. The next type of figurate number is the **hexagonal** number. (A hexagon has six sides.) Use the patterns above to predict the first five hexagonal numbers.

- **35.** Eight times any triangular number, plus 1, is a square number. Show that this is true for the first four triangular numbers.
- **36.** Divide the first triangular number by 3 and record the remainder. Divide the second triangular number by 3 and record the remainder. Repeat this procedure several more times. Do you notice a pattern?
- **37.** Repeat Exercise 36, but instead use square numbers and divide by 4. What pattern is determined?
- **38.** Exercises 36 and 37 are specific cases of the following: In general, when the numbers in the sequence of *n*-agonal numbers are divided by *n*, the sequence of remainders obtained is a repeating sequence. Verify this for n = 5 and n = 6.

39. Every square number can be written as the sum of two triangular numbers. For example, 16 = 6 + 10. This can be represented geometrically by dividing a square array of dots with a line as illustrated below.



The triangular arrangement above the line represents 6, the one below the line represents 10, and the whole arrangement represents 16. Show how the square numbers 25 and 36 may likewise be geometrically represented as the sum of two triangular numbers.

- **40.** A fraction is reduced to *lowest terms* if the greatest common factor of its numerator and its denominator is 1. For example, 3/8 is reduced to lowest terms, but 4/12 is not.
 - (a) For n = 2 to n = 8, form the fractions

*n*th square number

(n + 1)th square number

- (b) Repeat part (a), but use triangular numbers instead.
- (c) Use inductive reasoning to make a conjecture based on your results from parts (a) and (b), observing whether the fractions are reduced to lowest terms.
- **41.** Complete the following table.



Use your results to answer the following, using inductive reasoning.

(a) What kind of figurate number is obtained when you find the average of n^2 and n? (See Row C.)

- (b) If you square *n* and then subtract *n* from the result, and then divide by 2, what kind of figurate number is obtained? (See Row E.)
- **42.** Find the smallest integer *N* greater than 1 such that two different figurate numbers exist for *N*. What are they?

In addition to the formulas for T_n , S_n , and P_n shown in the text, the following formulas are true for **hexagonal** numbers (H), **heptagonal** numbers (Hp), and **octagonal** numbers (O):

$$H_n = \frac{n(4n-2)}{2} \qquad Hp_n = \frac{n(5n-3)}{2}$$
$$O_n = \frac{n(6n-4)}{2}.$$

Use these formulas to find each of the following.

- **43**. the sixteenth square number
- **44.** the eleventh triangular number
- 45. the ninth pentagonal number
- **46.** the seventh hexagonal number
- 47. the tenth heptagonal number
- 48. the twelfth octagonal number
- **49.** Observe the formulas given for H_n , Hp_n , and O_n , and use patterns and inductive reasoning to predict the formula for N_n , the *n*th **nonagonal** number. (A nonagon has 9 sides.) Then use the fact that the sixth nonagonal number is 111 to further confirm your conjecture.
- **50.** Use the result of Exercise 49 to find the tenth nonagonal number.

Use inductive reasoning to answer each question in Exercises 51-54.

- **51.** If you add two consecutive triangular numbers, what kind of figurate number do you get?
- **52.** If you add the squares of two consecutive triangular numbers, what kind of figurate number do you get?
- **53.** Square a triangular number. Square the next triangular number. Subtract the smaller result from the larger. What kind of number do you get?
- **54.** Choose a value of *n* greater than or equal to 2. Find T_{n-1} , multiply it by 3, and add *n*. What kind of figurate number do you get?