1.3 Strategies for Problem Solving

George Polya, author of the classic *How to Solve it,* died at the age of 97 on September 7, 1985. A native of Budapest, Hungary, he was once asked why there were so many good mathematicians to come out of Hungary at the turn of the century. He theorized that it was because mathematics is the cheapest science. It does not require any expensive equipment, only pencil and paper. He authored or coauthored more than 250 papers in many languages, wrote a number of books, and was a brilliant lecturer and teacher. Yet, interestingly enough, he never learned to drive a car.

In the first two sections of this chapter we stressed the importance of pattern recognition and the use of inductive reasoning in solving problems. There are other useful approaches. These ideas will be used throughout the text. (Problem solving is not a topic to be covered one day and then forgotten the next!)

Probably the most famous study of problem-solving techniques was developed by George Polya (1888–1985), among whose many publications was the modern classic *How to Solve It*. In this book, Polya proposed a four-step process for problem solving.

Polya's Four-Step Process for Problem Solving

1. **Understand the problem.** You cannot solve a problem if you do not understand what you are asked to find. The problem must be read and analyzed carefully. You will probably need to read it several times. After you have done so, ask yourself, "What must I find?"
2. **Devise a plan.** There are many ways to attack a problem and decide what plan is appropriate for the particular problem you are solving. See the list headed "Problem-Solving Strategies" for a number of possible approaches.
3. **Carry out the plan.** Once you know how to approach the problem, carry out your plan. You may run into "dead ends" and unforeseen roadblocks, but be persistent. If you are able to solve a problem without a struggle, it isn't much of a problem, is it?
4. **Look back and check.** Check your answer to see that it is reasonable. Does it satisfy the conditions of the problem? Have you answered all the questions the problem asks? Can you solve the problem a different way and come up with the same answer?

In Step 2 of Polya's problem-solving process, we are told to devise a plan. Here are some strategies that may prove useful.

Problem-Solving Strategies

Make a table or a chart.
Look for a pattern.
Solve a similar simpler problem.
Draw a sketch.
Use inductive reasoning.
Write an equation and solve it.
If a formula applies, use it.
Work backward.
Guess and check.
Use trial and error.
Use common sense.
Look for a "catch" if an answer seems too obvious or impossible.

A particular problem solution may involve one or more of the strategies listed here, and you should try to be creative in your problem-solving techniques. The examples that follow illustrate some of these strategies. As you read through them, keep in mind that it is one thing to read a problem solution or watch a teacher solve a problem, but it is another to be able to do it yourself. Have you ever watched your teacher solve a problem and said to yourself, "It looks easy when she does it, but I have trouble doing it myself"? Your teacher has spent many years practicing and studying problem-solving techniques. Like any skill, proficiency in problem solving requires perseverance and hard work.

FOR FURTHER THOUGHT

Various forms of the following problem have been around for many years.

In Farmer Jack's will, Jack bequeathed 1/2 of his horses to his son Johnny, 1/3 to his daughter Linda, and 1/9 to his son Jeff. Jack had 17 horses, so how were they to comply with the terms of the will? Certainly, horses cannot be divided up into fractions. Their attorney, Schwab, came to their rescue, and was able to execute the will to the satisfaction of all. How did she do it?

Here is the solution:

Schwab added one of her horses to the 17, giving a total of 18. Johnny received 1/2 of 18, or 9, Linda received 1/3 of 18, or 6, and Jeff received 1/9 of 18, or 2. That accounted for a total of $9 + 6 + 2 = 17$ horses. Then Schwab took back her horse, and everyone was happy.

For Group Discussion

Discuss whether the terms of the will were actually fulfilled, according to the letter of the law.

Fibonacci (1170–1250) discovered the sequence named after him in a problem on rabbits. Fibonacci (son of Bonaccio) is one of several names for Leonardo of Pisa. His father managed a warehouse in present-day Bougie (or Bejaia), in Algeria. Thus it was that Leonardo Pisano studied with a Moorish teacher and learned the "Indian" numbers that the Moors and other Moslems brought with them in their westward drive.

Fibonacci wrote books on algebra, geometry, and trigonometry.

EXAMPLE 1 ***Solving a Problem by Using a Table or a Chart*** A man put a pair of rabbits in a cage. During the first month the rabbits produced no offspring but each month thereafter produced one new pair of rabbits. If each new pair thus produced reproduces in the same manner, how many pairs of rabbits will there be at the end of one year?

This problem is a famous one in the history of mathematics and first appeared in *Liber Abaci*, a book written by the Italian mathematician Leonardo Pisano (also known as Fibonacci) in the year 1202. Let us apply Polya's process to solve it.

1. **Understand the problem.** After several readings, we can reword the problem as follows: How many pairs of rabbits will the man have at the end of one year if he starts with one pair, and they reproduce this way: During the first month of life, each pair produces no new rabbits, but each month thereafter each pair produces one new pair?
2. **Devise a plan.** Since there is a definite pattern to how the rabbits will reproduce, we can construct a table as shown below. Once the table is completed, the final entry in the final column is our answer.

Month	Number of Pairs at Start	Number of New Pairs Produced	Number of Pairs at End of Month
1st			
2nd			
3rd			
4th			
5th			
6th			
7th			
8th			
9th			
10th			
11th			
12th			

Problem solving should be the central focus of the mathematics curriculum. As such, it is a primary goal of all mathematics instruction and an integral part of all mathematical activity. Problem solving is not a distinct topic but a process that should permeate the entire program and provide the context in which concepts and skills can be learned.

From *Curriculum and Evaluation Standards for School Mathematics,* 1989 (National Council of Teachers of Mathematics)

3. **Carry out the plan.** At the start of the first month there is only one pair of rabbits. No new pairs are produced during the first month, so there is $1 + 0 = 1$ pair present at the end of the first month. This pattern continues throughout the table. We add the number in the first column of numbers to the number in the second column to get the number in the third.

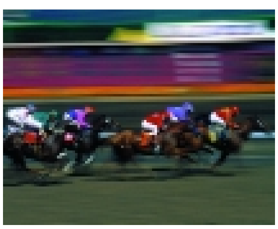
Month	Number of Pairs at Start	+	Number of New Pairs Produced	=	Number of Pairs at End of Month
1st	**1**		0		1
2nd	**1**		1		2
3rd	**2**		1		3
4th	**3**		2		5
5th	**5**		3		8
6th	**8**		5		13
7th	**13**		8		21
8th	**21**		13		34
9th	**34**		21		55
10th	**55**		34		89
11th	**89**		55		144
12th	**144**		89		233

$1 + 0 = 1$
$1 + 1 = 2$
$2 + 1 = 3$
.
.
.
$144 + 89 = 233$

There will be 233 pairs of rabbits at the end of one year.

4. **Look back and check.** This problem can be checked by going back and making sure that we have interpreted it correctly, which we have. Double-check the arithmetic. We have answered the question posed by the problem, so the problem is solved. ■

The sequence shown in color in the table in Example 1 is the Fibonacci sequence, and many of its interesting properties will be investigated in a later chapter. In the remaining examples of this section, we will use Polya's process but will not list the steps specifically as we did in Example 1.



EXAMPLE 2 ***Solving a Problem by Working Backward*** Rob Zwettler goes to the racetrack with his buddies on a weekly basis. One week he tripled his money, but then lost \$12. He took his money back the next week, doubled it, but then lost \$40. The following week he tried again, taking his money back with him. He quadrupled it, and then played well enough to take that much home with him, a total of \$224. How much did he start with the first week?

This problem asks us to find Rob's starting amount, given information about his winnings and losses. We also know his final amount. While we could write an algebraic equation to solve this problem, the method of working backward can be applied quite easily. Since his final amount was \$224 and this represents four times the amount he started with on the third week, we *divide* \$224 by 4 to find that he started the third week with \$56. Before he lost \$40 the second week, he had this \$56 plus the \$40 he lost, giving him \$96. This represented double what he started with, so he started with \$96 *divided by* 2, or \$48, the second week. Repeating this process once more for the first week, before his \$12 loss he had \$48 + \$12 = \$60, which represents triple what he started with. Therefore, he started with \$60 ÷ 3, or \$20.

To check our answer, $20, observe the following equations that depict the winnings and losses:

$$\begin{aligned}\text{First week: } & (3 \times \$20) - \$12 = \$60 - \$12 = \$48\\ \text{Second week: } & (2 \times \$48) - \$40 = \$96 - \$40 = \$56\\ \text{Third week: } & (4 \times \$56) = \$224. \quad \text{His final amount}\end{aligned}$$

YOU TRY IT

EXAMPLE 3 ***Solving a Problem by Trial and Error*** The mathematician Augustus De Morgan lived in the nineteenth century. He once made the following statement: "I was x years old in the year x^2." In what year was De Morgan born?

We must find the year of De Morgan's birth. The problem tells us that he lived in the nineteenth century, which is another way of saying that he lived during the 1800s. One year of his life was a perfect square, so we must find a number between 1800 and 1900 that is a perfect square. Use trial and error.

$$\begin{aligned}42^2 &= 1764\\ 43^2 &= 1849 \quad \text{1849 is between 1800 and 1900.}\\ 44^2 &= 1936\end{aligned}$$

The only natural number whose square is between 1800 and 1900 is 43, since $43^2 = 1849$. Therefore, De Morgan was 43 years old in 1849. The final step in solving the problem is to subtract 43 from 1849 to find the year of his birth: $1849 - 43 = 1806$. He was born in 1806.

While the following suggestion for a check may seem unorthodox, it does work: Look up De Morgan's birth date in a book dealing with mathematics history, such as *An Introduction to the History of Mathematics*, Sixth Edition, by Howard W. Eves.

Augustus De Morgan (see Example 3) was an English mathematician and philosopher, who served as professor at the University of London. He wrote numerous books, one of which was *A Budget of Paradoxes*. His work in set theory and logic led to laws that bear his name and are covered in other chapters. He died in the same year as Charles Babbage.

The next problem dates back to Hindu mathematics, circa 850.

EXAMPLE 4 ***Solving a Problem by Guessing and Checking*** One-fourth of a herd of camels was seen in the forest; twice the square root of that herd had gone to the mountain slopes; and 3 times 5 camels remained on the riverbank. What is the numerical measure of that herd of camels?

While an algebraic method of solving an equation could be used to solve this problem, we will show an alternative method. We are looking for a numerical measure of a herd of camels, so the number must be a counting number. Since the problem mentions "one-fourth of a herd" and "the square root of that herd," the number of camels must be both a multiple of 4 and a perfect square, so that no fractions will be encountered. The smallest counting number that satisfies both conditions is 4. Let us write an equation where x represents the numerical measure of the herd, and then substitute 4 for x to see if it is a solution.

$$\begin{array}{ccccccc}\text{"one-fourth of the herd"} & + & \text{"twice the square root of that herd"} & + & \text{"3 times 5 camels"} & = & \text{"the numerical measure of the herd"}\\ \frac{1}{4}x & + & 2\sqrt{x} & + & 3 \cdot 5 & = & x\end{array}$$

$$\begin{aligned}\frac{1}{4}(4) + 2\sqrt{4} + 3 \cdot 5 &= 4 \quad \text{Let } x = 4.\\ 1 + 4 + 15 &= 4 \quad ?\\ 20 &\neq 4\end{aligned}$$

Since 4 is not the solution, try 16, the next perfect square that is a multiple of 4.

$$\frac{1}{4}(\mathbf{16}) + 2\sqrt{\mathbf{16}} + 3 \cdot 5 = \mathbf{16} \qquad \text{Let } x = 16.$$

$$4 + 8 + 15 \stackrel{?}{=} 16$$

$$27 \neq 16$$

Since 16 is not a solution, try 36.

$$\frac{1}{4}(\mathbf{36}) + 2\sqrt{\mathbf{36}} + 3 \cdot 5 = \mathbf{36} \qquad \text{Let } x = 36.$$

$$9 + 12 + 15 \stackrel{?}{=} 36$$

$$36 = 36$$

We see that 36 is the numerical measure of the herd. Check in the words of the problem: "One-fourth of 36, plus twice the square root of 36, plus 3 times 5" gives 9 plus 12 plus 15, which equals 36.

EXAMPLE 5 ***Solving a Problem by Considering a Similar Simpler Problem and Looking for a Pattern*** The digit farthest to the right in a counting number is called the *ones* or *units* digit, since it tells how many ones are contained in the number when grouping by tens is considered. What is the ones (or units) digit in 2^{4000}?

Recall that 2^{4000} means that 2 is used as a factor 4000 times:

$$2^{4000} = \underbrace{2 \times 2 \times 2 \times \cdots \times 2}_{\text{4000 factors}}.$$

Certainly, we are not expected to evaluate this number. In order to answer the question, we can consider examining some smaller powers of 2 and then looking for a pattern. Let us start with the exponent 1 and look at the first twelve powers of 2.

$2^1 = \mathbf{2}$	$2^5 = 3\mathbf{2}$	$2^9 = 51\mathbf{2}$
$2^2 = \mathbf{4}$	$2^6 = 6\mathbf{4}$	$2^{10} = 102\mathbf{4}$
$2^3 = \mathbf{8}$	$2^7 = 12\mathbf{8}$	$2^{11} = 204\mathbf{8}$
$2^4 = 1\mathbf{6}$	$2^8 = 25\mathbf{6}$	$2^{12} = 409\mathbf{6}$

Notice that in each of the four rows above, the ones digit is the same. The final row, which contains the exponents 4, 8, and 12, has the ones digit 6. Each of these exponents is divisible by 4, and since 4000 is divisible by 4, we can use inductive reasoning to predict that the units digit in 2^{4000} is 6. (*Note:* The units digit for any other power can be found if we divide the exponent by 4 and consider the remainder. Then compare the result to the list of powers above. For example, to find the units digit of 2^{543}, divide 543 by 4 to get a quotient of 135 and a remainder of 3. The units digit is the same as that of 2^3, which is 8.)

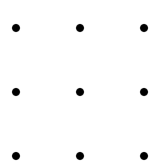


FIGURE 6

EXAMPLE 6 ***Solving a Problem by Drawing a Sketch*** An array of nine dots is arranged in a 3×3 square, as shown in Figure 6. Is it possible to join the dots with exactly four straight lines if you are not allowed to pick up your pencil from the paper and may not trace over a line that has already been drawn? If so, show how.

Figure 7 shows three attempts. In each case, something is wrong. In the first sketch, one dot is not joined. In the second, the figure cannot be drawn without picking up your pencil from the paper or tracing over a line that has already been drawn.

In the third figure, all dots have been joined, but you have used five lines as well as retraced over the figure.

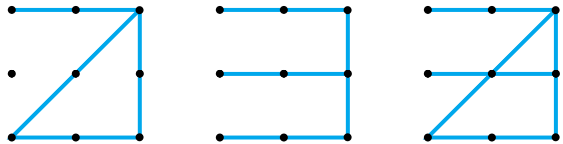


FIGURE 7

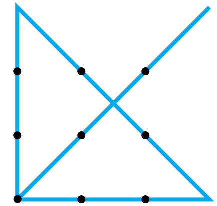


FIGURE 8

However, the conditions of the problem can be satisfied, as shown in Figure 8. We "went outside of the box," which was not prohibited by the conditions of the problem. This is an example of creative thinking—we used a strategy that is usually not considered at first, since our initial attempts involved "staying within the confines" of the figure.

The final example falls into a category of problems that involve a "catch" in the wording. Such problems often seem impossible to solve at first, due to preconceived notions about the matter being discussed. Some of these problems seem too easy or perhaps impossible at first, because we tend to overlook an obvious situation. We should look carefully at the use of language in such problems. And, of course, we should never forget to use common sense.

Photo not available

EXAMPLE 7 ***Solving a Problem Using Common Sense*** Two currently minted United States coins together have a total value of $1.05. One is not a dollar. What are the two coins?

Our initial reaction might be, "The only way to have two such coins with a total of $1.05 is to have a nickel and a dollar, but the problem says that one of them is not a dollar." This statement is indeed true. What we must realize here is that the one that is not a dollar is the nickel, and the *other* coin is a dollar! So the two coins are a dollar and a nickel.