# 4.1 Historical Numeration Systems

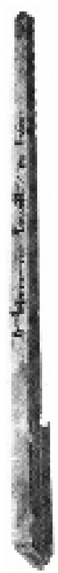
Primitive societies have little need for large numbers. Even today, the languages of some cultures contain no words for numbers beyond “one,” “two,” and maybe an indefinite word suggesting “many.” For example, according to UCLA physiologist Jared Diamond (*Discover,* Aug. 1987, p. 38), there are Gimi villages in New Guinea that use just two root words—*iya* for one and *rarido* for two. Slightly larger numbers are indicated using combinations of these two: for example, *rarido-rarido* is four and *rarido-rarido-iya* is five.

A practical method of keeping accounts by matching may have developed as humans established permanent settlements and began to grow crops and raise livestock. People might have kept track of the number of sheep in a flock by matching pebbles with the sheep, for example. The pebbles could then be kept as a record of the number of sheep.

A more efficient method is to keep a **tally stick.** With a tally stick, one notch or **tally** is made on a stick for each sheep. Tally sticks and tally marks have been found that appear to be many thousands of years old. Tally marks are still used today: for example, nine items are tallied by writing 𝍸 ||||.

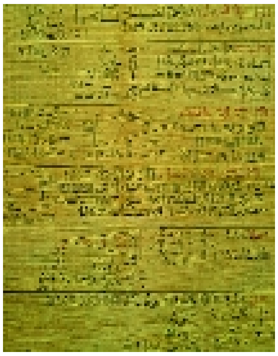
Tally sticks and groups of pebbles were an important advance in counting. By these methods, the idea of *number* began to develop. Early people began to see that a group of three chickens and a group of three dogs had something in common: the idea of *three*. Gradually, people began to think of numbers separately from the things they represented. Words and symbols were developed for various numbers.

The numerical records of ancient people give us some idea of their daily lives and create a picture of them as producers and consumers. For example, Mary and Joseph went to Bethlehem to be counted in a census—a numerical record. Even earlier than that, as long as 5000 years ago, the Egyptian and Sumerian peoples were using large numbers in their government and business records. Ancient documents reveal some of their numerical methods, as well as those of the Greeks, Romans, Chinese, and Hindus. Numeration systems became more sophisticated as the need arose.



**Tally sticks** like this one were used by the English in about 1400 A.D. to keep track of financial transactions. Each notch stands for one pound sterling.

**Ancient Egyptian Numeration—Simple Grouping** Early matching and tallying led to the essential ingredient of all more advanced numeration systems, that of **grouping.** Grouping allows for less repetition of symbols and also makes numerals easier to interpret. Most historical systems, including our own, have used groups of ten, indicating that people commonly learn to count by using their fingers. The size of the groupings (again, usually ten) is called the **base** of the number system. Bases of five, twenty, and sixty have also been used.



Much of our knowledge of **Egyptian mathematics** comes from the Rhind papyrus, from about 3800 years ago. A small portion of this papyrus, showing methods for finding the area of a triangle, is reproduced here.

The ancient Egyptian system is an example of a simple grouping system. It utilized ten as its base, and its various symbols are shown in Table 1. The symbol for 1 (|) is repeated, in a tally scheme, for 2, 3, and so on up to 9. A new symbol is introduced for 10 (∩), and that symbol is repeated for 20, 30, and so on, up to 90. This pattern enabled the Egyptians to express numbers up to 9,999,999 with just the seven symbols shown in the table.

**TABLE 1** Early Egyptian Symbols

| Number | Symbol | Description |
|---|---|---|
| 1 | 𓏺 | Stroke |
| 10 | 𓎆 | Heel bone |
| 100 | 𓍢 | Scroll |
| 1000 | 𓆼 | Lotus flower |
| 10,000 | 𓂭 | Pointing finger |
| 100,000 | 𓆐 | Burbot fish |
| 1,000,000 | 𓁨 | Astonished person |

The Egyptian symbols used denote the various **powers** of the base (ten):

$$10^0 = 1,\ 10^1 = 10,\ 10^2 = 100,\ 10^3 = 1000,\ 10^4 = 10{,}000,$$
$$10^5 = 100{,}000,\ \text{and}\ 10^6 = 1{,}000{,}000.$$

The smaller numerals at the right of the 10s, and slightly raised, are called **exponents.**

**EXAMPLE 1** Write in our system the number below.

𓆐𓆐𓆼𓆼𓆼𓆼𓆼𓍢𓍢𓍢𓍢∩∩∩∩∩∩∩∩∩|||||||

Refer to Table 1 for the values of the Egyptian symbols. Each 𓆐 represents 100,000. Therefore, two 𓆐s represent $2 \times 100{,}000$, or 200,000. Proceed as follows:

| | | | |
|---|---|---|---|
| two | 𓆐 | $2 \times 100{,}000 =$ | 200,000 |
| five | 𓆼 | $5 \times 1000 =$ | 5000 |
| four | 𓍢 | $4 \times 100 =$ | 400 |
| nine | ∩ | $9 \times 10 =$ | 90 |
| seven | \| | $7 \times 1 =$ | 7 |
| | | | 205,497. |

The number is 205,497.

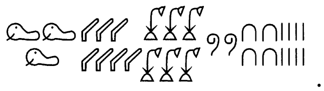
**Applied Mathematics** An Egyptian tomb painting shows scribes tallying the count of a grain harvest. Egyptian mathematics was oriented more to practicality than was Greek or Babylonian mathematics, although the Egyptians did have a formula for finding the volume of a certain portion of a pyramid.

| Number | Symbol |
|---|---|
| 1 | I |
| 5 | V |
| 10 | X |
| 50 | L |
| 100 | C |
| 500 | D |
| 1000 | M |

**Roman numerals** still appear today, mostly for decorative purposes: on clock faces, for chapter numbers in books, and so on. The system is essentially base ten, simple grouping, but with separate symbols for the intermediate values 5, 50, and 500, as shown above. If I is positioned left of V or X, it is subtracted rather than added. Likewise for X appearing left of L or C, and for C appearing left of D or M. Thus, for example, whereas CX denotes 110, XC denotes 90.

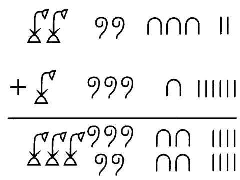
**EXAMPLE 2** Write 376,248 in Egyptian symbols.

Writing this number requires three 𓆐s, seven 𓂭s, six 𓆼s, two 𓍢s, four 𓎆s, and eight 𓏺s, or



Notice that the position or order of the symbols makes no difference in a simple grouping system. Each of the numbers 𓍢𓍢𓎆𓎆𓎆𓏺𓏺𓏺𓏺, 𓏺𓏺𓏺𓏺𓎆𓎆𓎆𓍢𓍢, and 𓏺𓏺𓎆𓎆𓍢𓍢𓎆𓏺𓏺 would be interpreted as 234. The most common order, however, is that shown in Examples 1 and 2, where like symbols are grouped together and groups of higher-valued symbols are positioned to the left.

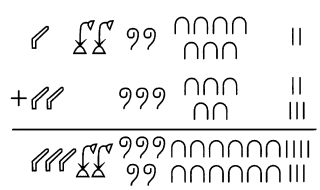
A simple grouping system is well suited to addition and subtraction. For example, to add 𓆼𓆼𓍢𓍢𓎆𓎆𓎆𓏺𓏺 and 𓆼𓍢𓍢𓍢𓎆𓏺𓏺𓏺𓏺𓏺𓏺 in the early Egyptian system, work as shown. Two 𓏺s plus six 𓏺s equal to eight 𓏺s, and so on.



Sum:

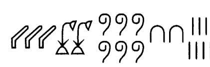
While we used a + sign for convenience and drew a line under the numbers, the Egyptians did not do this.

Sometimes regrouping, or "carrying," is needed as in the example below in which the answer contains more than nine heel bones. To regroup, get rid of ten heel bones from the tens group. Compensate for this by placing an extra scroll in the hundreds group.



Sum:

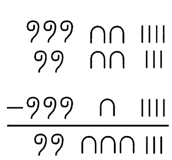
Regrouped answer:



Subtraction is done in much the same way, as shown in the next example.

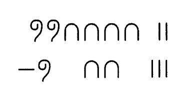
**EXAMPLE 3** Subtract in each of the following.

**(a)**



Difference:

**(b)**



In part (b), to subtract four 𓏺s from two 𓏺s, "borrow" one heel bone, which is equivalent to ten 𓏺s. Finish the problem after writing ten additional 𓏺s on the right.

| Greek Numerals | | | |
|---|---|---|---|
| 1 | α | 60 | ξ |
| 2 | β | 70 | ο |
| 3 | γ | 80 | π |
| 4 | δ | 90 | φ |
| 5 | ε | 100 | ρ |
| 6 | ϛ | 200 | σ |
| 7 | ζ | 300 | τ |
| 8 | η | 400 | υ |
| 9 | θ | 500 | ϕ |
| 10 | ι | 600 | χ |
| 20 | κ | 700 | ψ |
| 30 | λ | 800 | ω |
| 40 | μ | 900 | ϡ |
| 50 | ν | | |

**What About the Greeks?**
Classical Greeks used letters of their alphabet as numerical symbols. The base of the system was the number 10, and numbers 1 through 9 were symbolized by the first nine letters of the alphabet. Rather than using repetition or multiplication, they assigned nine more letters to multiples of 10 (through 90) and more letters to multiples of 100 (through 900). This is called a ciphered system, and it sufficed for small numbers. For example, 57 would be $\nu\zeta$; 573 would be $\phi o\gamma$; and 803 would be $\omega\gamma$. A small stroke was used with a units symbol for multiples of 1000 (up to 9000); thus 1000 would be ,$\alpha$ or $'\alpha$. Often M would indicate tens of thousands (M for myriad = 10,000) with the multiples written above M.

Regroupeded: 𓍢𓍢 ∩∩∩ |||||| ||||||

− 𓍢 ∩∩ ||||

Difference: 𓍢 ∩ ||||||||

A procedure such as those described above is called an **algorithm:** a rule or method for working a problem. The Egyptians used an interesting algorithm for multiplication that requires only an ability to add and to double numbers, as shown in Example 4. For convenience, this example uses our symbols rather than theirs.

**EXAMPLE 4** A stone used in building a pyramid has a rectangular base measuring 5 by 18 cubits. Find the area of the base.

The area of a rectangle is found by multiplying the length and the width; in this problem, we must find $5 \times 18$. To begin, build two columns of numbers, as shown below. Start the first column with 1, and the second column with 18. Each column is built downward by doubling the number above. Keep going until the first column contains numbers that can be added to equal 5. Here $1 + 4 = 5$. To find $5 \times 18$, add only those numbers from the second column that correspond to 1 and 4. Here 18 and 72 are added to get the answer 90. The area of the base of the stone is 90 square cubits.

$$1 + 4 = 5 \left\{ \begin{array}{lll} \rightarrow \mathbf{1} & \mathbf{18} & \leftarrow \text{Corresponds to 1} \\ \quad 2 & 36 & \\ \rightarrow \mathbf{4} & \mathbf{72} & \leftarrow \text{Corresponds to 4} \end{array} \right\} 18 + 72 = 90$$

Finally, $5 \times 18 = 90$.

**EXAMPLE 5** Use the Egyptian multiplication algorithm to find $19 \times 70$.

YOU TRY IT

$$\begin{array}{rr} \rightarrow \mathbf{1} & \mathbf{70} \leftarrow \\ \rightarrow \mathbf{2} & \mathbf{140} \leftarrow \\ 4 & 280 \\ 8 & 560 \\ \rightarrow \mathbf{16} & \mathbf{1120} \leftarrow \end{array}$$

Form two columns, headed by 1 and by 70. Keep doubling until there are numbers in the first column that add up to 19. (Here, $1 + 2 + 16 = 19$.) Then add corresponding numbers from the second column: $70 + 140 + 1120 = 1330$, so that $19 \times 70 = 1330$.

## Traditional Chinese Numeration—Multiplicative Grouping

Examples 1 through 3 above show that simple grouping, although an improvement over tallying, still requires considerable repetition of symbols. To denote 90, for example, the ancient Egyptian system must utilize nine ∩s: ∩∩∩∩∩∩∩∩∩. If an additional symbol (a "multiplier") was introduced to represent nine, say "9," then 90 could be denoted 9 ∩. All possible numbers of repetitions of powers of the base could be handled by introducing a separate multiplier symbol for each counting number less than the base. Although the ancient Egyptian system apparently did not evolve in this direction, just such a system was developed many years ago in China.

**TABLE 2**

| Number | Symbol |
|---|---|
| 1 | 一 |
| 2 | 二 |
| 3 | 三 |
| 4 | 四 |
| 5 | 五 |
| 6 | 六 |
| 7 | 七 |
| 8 | 八 |
| 9 | 九 |
| 10 | 十 |
| 100 | 百 |
| 1000 | 千 |
| 0 | 零 |

This photo is of a **quipu.** In *Ethnomathematics: A Multicultural View of Mathematical Ideas,* Marcia Ascher writes:

*A quipu is an assemblage of colored knotted cotton cords. Among the Inca, cotton cloth and cordage were of great importance. Used to construct bridges, in ceremonies, for tribute, and in every phase of the life cycle from birth to death, cotton cordage and cloth were of unparalleled importance in Inca culture and, hence, not a surprising choice for its principal medium. The colors of the cords, the way the cords are connected, the relative placement of the cords, the spaces between the cords, the types of knots on the individual cords, and the relative placement of the knots are all part of the logical-numerical recording.*

It was later adopted, for the most part, by the Japanese, with several versions occurring over the years. Here we show the predominant Chinese version, which used the symbols shown in Table 2. We call this type of system a **multiplicative grouping** system. In general, such a system would involve pairs of symbols, each pair containing a multiplier (with some counting number value less than the base) and then a power of the base. The Chinese numerals are read from top to bottom rather than from left to right.

Three features distinguish this system from a strictly pure multiplicative grouping system. First, the number of 1s is indicated using a single symbol rather than a pair. In effect, the multiplier $(1, 2, 3, \ldots, 9)$ is written but the power of the base $(10^0)$ is not. Second, in the pair indicating 10s, if the multiplier is 1, then that multiplier is omitted. Just the symbol for 10 is written. Third, when a given power of the base is totally missing in a particular number, this omission is shown by the inclusion of the special zero symbol. (See Table 2.) If two or more consecutive powers are missing, just one zero symbol serves to note the total omission. The omission of 1s and 10s, and any other powers occurring at the extreme bottom of a numeral, need not be noted with a zero symbol. (Note that, for clarification in the examples that follow, we have emphasized the grouping into pairs by spacing and by using braces. These features are *not* part of the actual numeral.)

**EXAMPLE 6** Interpret the Chinese numerals below.

**(a)**

| | |
|---|---|
| 三 千 | $3 \times 1000 = 3000$ |
| 一 百 | $1 \times 100 = 100$ |
| 六 十 | $6 \times 10 = 60$ |
| 四 | $4(\times 1) = 4$ |
| | Total: 3164 |

**(b)**

| | |
|---|---|
| 七 百 | $7 \times 100 = 700$ |
| 零 | $0(\times 10) = 00$ |
| 三 | $3(\times 1) = 3$ |
| | Total: 703 |

**(c)**

| | |
|---|---|
| 五 千 | $5 \times 1000 = 5000$ |
| 零 | $0(\times 100) = 000$<br>$0(\times 10) = 00$ |
| 九 | $9(\times 1) = 9$ |
| | Total: 5009 |

**(d)**

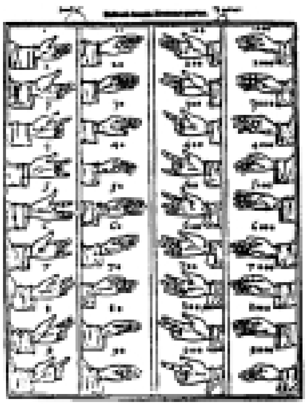
| | |
|---|---|
| 四 千 | $4 \times 1000 = 4000$ |
| 二 百 | $2 \times 100 = 200$ |
| | Total: 4200 |

**EXAMPLE 7** Write Chinese numerals for these numbers.

**(a)** 614

This number is made up of six 100s, one 10, and one 4, as depicted at the right.

| | |
|---|---|
| $6 \times 100$: | 六 百 |
| $(1 \times)10$: | 十 |
| $4(\times 1)$: | 四 |



**Finger Reckoning** There is much evidence that early humans (in various cultures) used their fingers to represent numbers. As calculations became more complicated, *finger reckoning,* as shown in this sketch, became popular. The Romans became adept at this sort of calculating, carrying it to 10,000 or perhaps higher.

| Number | Symbol |
|---|---|
| 1 | 𒁹 |
| 10 | 𒌋 |

**Babylonian numeration** was positional, base sixty. But the face values within the positions were base ten simple grouping numerals, formed with the two symbols shown above. (These symbols resulted from the Babylonian method of writing on clay with a wedge-shaped stylus.) The numeral

𒌋𒌋𒁹𒁹𒁹𒌋𒌋𒌋𒌋𒁹

denotes 1421 ($23 \times 60 + 41 \times 1$).

**(b)** 5090

The number consists of five 1000s, no 100s, and nine 10s (no 1s).

$5 \times 1000$: 五 千

$0(\times 100)$: 零

$9 \times 10$: 九 十

## Hindu-Arabic Numeration—Positional System

A simple grouping system relies on repetition of symbols to denote the number of each power of the base. A multiplicative grouping system uses multipliers in place of repetition, which is more efficient. But the ultimate in efficiency is attained only when we proceed to the next step, a **positional** system, in which only the multipliers are used. The various powers of the base require no separate symbols, since the power associated with each multiplier can be understood by the position that the multiplier occupies in the numeral. If the Chinese system had evolved into a positional system, then the numeral for 7482 could be written

七<br>四<br>八<br>二 rather than 七<br>千<br>四<br>百<br>八<br>十<br>二.

The lowest symbol is understood to represent two 1s ($10^0$), the next pair up denotes eight 10s ($10^1$), then four 100s ($10^2$), and finally seven 1000s ($10^3$). Each symbol in a numeral now has both a *face value,* associated with that particular symbol (the multiplier value), and a *place value* (a power of the base), associated with the place, or position, occupied by the symbol.

### Positional Numeration

In a positional numeral, each symbol (called a **digit**) conveys two things:

1. **face value**—the inherent value of the symbol
2. **place value**—the power of the base which is associated with the position that the digit occupies in the numeral.

The place values in a Hindu-Arabic numeral, from right to left, are 1, 10, 100, 1000, and so on. The three 4s in the number 46,424 all have the same face value but different place values. The first 4, on the left, denotes four 10,000s, the next one denotes four 100s, and the one on the right denotes four 1s. Place values (in base ten) are named as shown here:

| Billions, | Hundred millions | Ten millions | Millions, | Hundred thousands | Ten thousands | Thousands, | Hundreds | Tens | Units | Decimal point. |
|---|---|---|---|---|---|---|---|---|---|---|
| 8, | 3 | 2 | 1, | 4 | 5 | 6, | 7 | 9 | 5 | . |

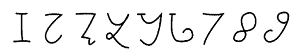
This numeral is read as eight billion, three hundred twenty-one million, four hundred fifty-six thousand, seven hundred ninety-five.

**From Tally to Tablet** The clay tablet above, despite damage, shows the durability of the mud of the Babylonians. Thousands of years after the tablets were made, Babylonian algebra problems can be worked out from the original writings.

In recent times, herdsmen make small "tokens" out of this clay as tallies of animals. Similar tokens, some 10,000 years old, have been unearthed by archaeologists in the land that was once Babylonia. Shaped like balls, disks, cones, and other regular forms, they rarely exceed 5 cm in diameter.

To work successfully, a positional system must have a symbol for zero to serve as a **placeholder** in case one or more powers of the base are not needed. Because of this requirement, some early numeration systems took a long time to evolve to a positional form, or never did. Although the traditional Chinese system does utilize a zero symbol, it never did incorporate all the features of a positional system, but remained essentially a multiplicative grouping system.

The one numeration system that did achieve the maximum efficiency of positional form is our own system, commonly known for historical reasons as the **Hindu-Arabic** system. It was developed over many centuries. Its symbols have been traced to the Hindus of 200 B.C. They were picked up by the Arabs and eventually transmitted to Spain, where a late tenth-century version appeared like this:



The earliest stages of the system evolved under the influence of navigational, trade, engineering, and military requirements. And in early modern times, the advance of astronomy and other sciences led to a structure well suited to fast and accurate computation. The purely positional form that the system finally assumed was introduced to the West by Leonardo Fibonacci of Pisa (1170–1250) early in the thirteenth century. But widespread acceptance of standardized symbols and form was not achieved until the invention of printing during the fifteenth century. Since that time, no better system of numeration has been devised, and the positional base ten Hindu-Arabic system is commonly used around the world today. (In India, where it all began, standardization still is not totally achieved, as various local systems are used today.)

In the next section we shall look in more detail at the structure of the Hindu-Arabic system and some early methods and devices for doing computation.