4.2 Arithmetic in the Hindu-Arabic System

The historical development of numeration culminated in positional systems, the most successful of which is the Hindu-Arabic system. As stated in the previous section, Hindu-Arabic place values are powers of the base ten. For example, 10^4 denotes the fourth power of ten. Such expressions are often called **exponential expressions.** In this case, 10 is the **base** and 4 is the **exponent.** Exponents actually indicate repeated multiplication of the base:

$$10^4 = \underbrace{10 \times 10 \times 10 \times 10}_{\text{Four factors of 10}} = 10{,}000.$$

In the same way, $10^2 = 10 \times 10 = 100$, $10^6 = 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 1{,}000{,}000$, and so on. The base does not have to be 10; for example,

$$4^3 = 4 \times 4 \times 4 = 64, \quad 2^2 = 2 \times 2 = 4,$$
$$3^5 = 3 \times 3 \times 3 \times 3 \times 3 = 243,$$

and so on. Expressions of this type are defined in general as follows.

Digits This Iranian stamp should remind us that counting on fingers (and toes) is an age-old practice. In fact, our word *digit,* referring to the numerals 0–9, comes from a Latin word for "finger" (or "toe"). Aristotle first noted the relationships between fingers and base ten in Greek numeration. Anthropologists go along with the notion. Some cultures, however, have used two, three, or four as number bases, for example, counting on the joints of the fingers or the spaces between them.

Exponential Expressions

For any number a and any counting number m,

$$a^m = \underbrace{a \times a \times a \times \cdots \times a}_{m \text{ factors of } a}.$$

The number a is the **base,** m is the **exponent,** and a^m is read "a to the power m."

EXAMPLE 1 Find each power.

(a) $10^3 = 10 \times 10 \times 10 = 1000$

(10^3 is read "10 cubed," or "10 to the third power.")

(b) $7^2 = 7 \times 7 = 49$

(7^2 is read "7 squared," or "7 to the second power.")

(c) $5^4 = 5 \times 5 \times 5 \times 5 = 625$

(5^4 is read "5 to the fourth power.") ■

To simplify work with exponents, it is agreed that $a^0 = 1$ for any nonzero number a. By this agreement, $7^0 = 1$, $52^0 = 1$, and so on. At the same time, $a^1 = a$ for any number a. For example, $8^1 = 8$, and $25^1 = 25$. The exponent 1 is usually omitted.

With the use of exponents, numbers can be written in **expanded form** in which the value of the digit in each position is made clear. For example, write 924 in expanded form by thinking of 924 as nine 100s plus two 10s plus four 1s, or

$$924 = 900 + 20 + 4$$
$$924 = (9 \times 100) + (2 \times 10) + (4 \times 1).$$

By the definition of exponents, $100 = 10^2$, $10 = 10^1$, and $1 = 10^0$. Use these exponents to write 924 in expanded form as follows:

$$924 = (9 \times 10^2) + (2 \times 10^1) + (4 \times 10^0).$$

VIDEO

EXAMPLE 2 Write each number in expanded form.

(a) $1906 = (1 \times 10^3) + (9 \times 10^2) + (0 \times 10^1) + (6 \times 10^0)$

Since $0 \times 10^1 = 0$, this term could be omitted, but the form is clearer with it included.

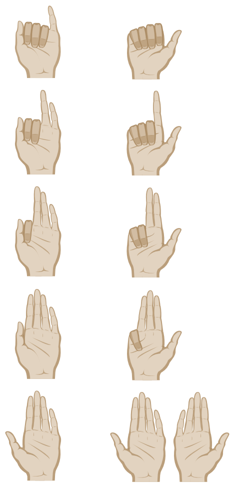
VIDEO

(b) $46{,}424 = (4 \times 10^4) + (6 \times 10^3) + (4 \times 10^2) + (2 \times 10^1) + (4 \times 10^0)$ ■

EXAMPLE 3 Simplify each of the following expansions.

(a) $(3 \times 10^5) + (2 \times 10^4) + (6 \times 10^3) + (8 \times 10^2) + (7 \times 10^1) + (9 \times 10^0) = 326{,}879$

(b) $(2 \times 10^1) + (8 \times 10^0) = 28$ ■



Finger Counting The first digits many people used for counting were their fingers. In Africa the Zulu used the method shown here to count to ten. They started on the left hand with palm up and fist closed. The Zulu finger positions for 1–5 are shown above on the left. The Zulu finger positions for 6–10 are shown on the right.

Expanded notation can be used to see why standard algorithms for addition and subtraction really work. The key idea behind these algorithms is based on the **distributive property,** which will be discussed more fully later in this chapter. It can be written in one form as follows.

Distributive Property

For all real numbers a, b, and c,

$$(b \times a) + (c \times a) = (b + c) \times a.$$

For example,

$$(3 \times 10^4) + (2 \times 10^4) = (3 + 2) \times 10^4$$
$$= 5 \times 10^4.$$

EXAMPLE 4 Use expanded notation to add 23 and 64.

$$\begin{aligned} 23 &= (2 \times 10^1) + (3 \times 10^0) \\ +64 &= (6 \times 10^1) + (4 \times 10^0) \\ & \quad (8 \times 10^1) + (7 \times 10^0) = 87 \end{aligned}$$

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Subtraction works in much the same way.

EXAMPLE 5 Find $695 - 254$.

$$\begin{aligned} 695 &= (6 \times 10^2) + (9 \times 10^1) + (5 \times 10^0) \\ -254 &= (2 \times 10^2) + (5 \times 10^1) + (4 \times 10^0) \\ & \quad (4 \times 10^2) + (4 \times 10^1) + (1 \times 10^0) = 441 \end{aligned}$$

Expanded notation and the distributive property can also be used to show how to solve addition problems where a power of 10 ends up with a multiplier of more than one digit.

EXAMPLE 6 Use expanded notation to add 75 and 48.

$$\begin{aligned} 75 &= \ (7 \times 10^1) + \ (5 \times 10^0) \\ +48 &= \ (4 \times 10^1) + \ (8 \times 10^0) \\ & \quad (11 \times 10^1) + (13 \times 10^0) \end{aligned}$$

Since the units position (10^0) has room for only one digit, 13×10^0 must be modified:

$$13 \times 10^0 = (10 \times 10^0) + (3 \times 10^0) \qquad \text{Distributive property}$$
$$= (1 \times 10^1) + (3 \times 10^0).$$

In effect, the 1 from 13 moved to the left from the units position to the tens position. This is called "carrying." Now our sum is

$$
\begin{aligned}
&\underbrace{(11 \times 10^1) + (1 \times 10^1)} + (3 \times 10^0) \\
&= (12 \times 10^1) + (3 \times 10^0) && \text{Distributive property} \\
&= (10 \times 10^1) + (2 \times 10^1) + (3 \times 10^0) \\
&= (1 \times 10^2) + (2 \times 10^1) + (3 \times 10^0) \\
&= 123.
\end{aligned}
$$

Subtraction problems often require "borrowing," which can also be clarified with expanded notation.

EXAMPLE 7 Use expanded notation to subtract 186 from 364.

$$
\begin{aligned}
364 &= (3 \times 10^2) + (6 \times 10^1) + (4 \times 10^0) \\
\underline{-186} &= \underline{(1 \times 10^2) + (8 \times 10^1) + (6 \times 10^0)}
\end{aligned}
$$

Since, in the units position, we cannot subtract 6 from 4, we modify the top expansion as follows (the units position borrows from the tens position):

$$
\begin{aligned}
&(3 \times 10^2) + \underbrace{(6 \times 10^1)} + (4 \times 10^0) \\
&= (3 \times 10^2) + (5 \times 10^1) + (1 \times 10^1) + (4 \times 10^0) && \text{Distributive property} \\
&= (3 \times 10^2) + (5 \times 10^1) + \underbrace{(10 \times 10^0) + (4 \times 10^0)} \\
&= (3 \times 10^2) + (5 \times 10^1) + (14 \times 10^0). && \text{Distributive property}
\end{aligned}
$$

(We can now subtract 6 from 14 in the units position, but cannot take 8 from 5 in the tens position, so we continue the modification, borrowing from the hundreds to the tens position.)

$$
\begin{aligned}
&\underbrace{(3 \times 10^2)} + (5 \times 10^1) + (14 \times 10^0) \\
&= (2 \times 10^2) + (1 \times 10^2) + (5 \times 10^1) + (14 \times 10^0) && \text{Distributive property} \\
&= (2 \times 10^2) + \underbrace{(10 \times 10^1) + (5 \times 10^1)} + (14 \times 10^0) \\
&= (2 \times 10^2) + (15 \times 10^1) + (14 \times 10^0) && \text{Distributive property}
\end{aligned}
$$

Now we can complete the subtraction.

$$
\begin{array}{rrrr}
 & (2 \times 10^2) + & (15 \times 10^1) + & (14 \times 10^0) \\
- & (1 \times 10^2) + & (8 \times 10^1) + & (6 \times 10^0) \\
\hline
 & (1 \times 10^2) + & (7 \times 10^1) + & (8 \times 10^0) = 178
\end{array}
$$

Examples 4 through 7 used expanded notation and the distributive property to clarify our usual addition and subtraction methods. In practice, our actual work for these four problems would appear as follows:

$$
\begin{array}{r}
23 \\
+\ 64 \\
\hline
87
\end{array}
\qquad
\begin{array}{r}
695 \\
-254 \\
\hline
441
\end{array}
\qquad
\begin{array}{r}
\scriptstyle 1 \\[-4pt]
75 \\
+\ 48 \\
\hline
123
\end{array}
\qquad
\begin{array}{r}
\scriptstyle 2\ \ 15 \\[-4pt]
\not{3}\ \not{6}\,{}^{1}4 \\
-1\ 8\ 6 \\
\hline
1\ 7\ 8.
\end{array}
$$

The ***Carmen de Algorismo*** (opening verses shown here) by Alexander de Villa Dei, thirteenth century, popularized the new art of "algorismus":

> *. . . from these twice five figures 0 9 8 7 6 5 4 3 2 1 of the Indians we benefit . . .*

The *Carmen* related that Algor, an Indian king, invented the art. But actually, "algorism" (or "algorithm") comes in a roundabout way from the name Muhammad ibn Musa al-Khorârizmi, an Arabian mathematician of the ninth century, whose arithmetic book was translated into Latin. Furthermore, this Muhammad's book on equations, *Hisab al-jabr w'almuqâbalah,* yielded the term "algebra" in a similar way.

Palmtop computers are the latest in the development of calculating devices.

The procedures seen in this section also work for positional systems with bases other than ten.

Since our numeration system is based on powers of ten, it is often called the **decimal system,** from the Latin word *decem*, meaning ten.* Over the years, many methods have been devised for speeding calculations in the decimal system. One of the oldest is the **abacus,** a device made with a series of rods with sliding beads and a dividing bar. Reading from right to left, the rods have values of 1, 10, 100, 1000, and so on. The bead above the bar has five times the value of those below. Beads moved *toward* the bar are in the "active" position, and those toward the frame are ignored. In our illustrations of abaci (plural form of abacus), such as in Figure 1, the activated beads are shown in black for emphasis.

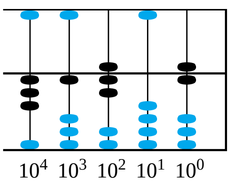


FIGURE 1

EXAMPLE 8 The number on the abacus in Figure 1 is found as follows:

$$\begin{aligned}&(3 \times 10{,}000) + (1 \times 1000) + [(1 \times 500) + (2 \times 100)] + 0 \\ &\quad + [(1 \times 5) + (1 \times 1)] \\ &= 30{,}000 + 1000 + 500 + 200 + 0 + 5 + 1 \\ &= 31{,}706.\end{aligned}$$

As paper became more readily available, people gradually switched from devices like the abacus (though these still are commonly used in some areas) to paper-and-pencil methods of calculation. One early scheme, used both in India and Persia, was the **lattice method,** which arranged products of single digits into a diagonalized lattice, as shown in the following example.

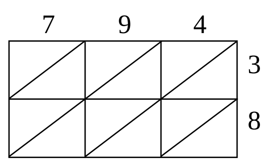
Merry Math The following two rhymes come from *Marmaduke Multiply's Merry Method of Making Minor Mathematicians,* a primer published in the late 1830s in Boston:

1. Twice 2 are 4. Pray hasten on before.
2. Five times 5 are 25. I thank my stars I'm yet alive.

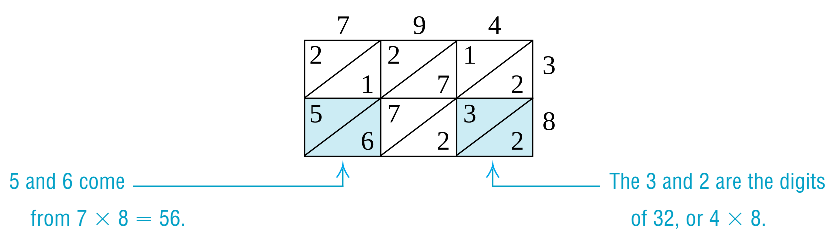
EXAMPLE 9 Find the product 38 × 794 by the lattice method.

Work as follows.

Step 1: Write the problem, with one number at the side and one across the top.

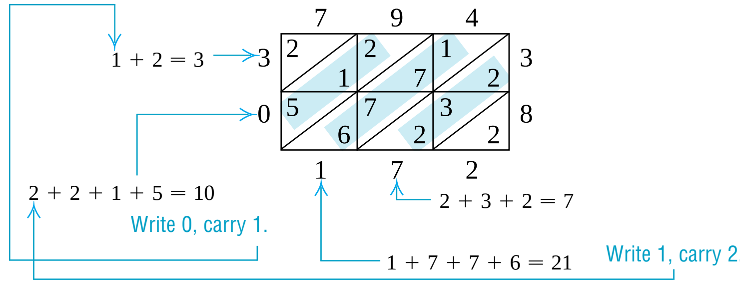


Step 2: Within the lattice, write the products of all pairs of digits from the top and side.



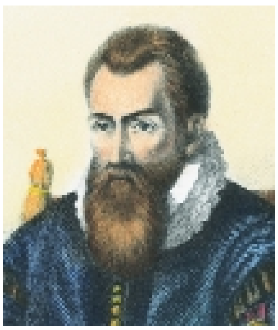
**December* was the tenth month in an old form of the calendar. It is interesting to note that *decem* became *dix* in the French language; a ten-dollar bill, called a "dixie," was in use in New Orleans before the Civil War. "Dixie Land" was a nickname for that city before Dixie came to refer to all the Southern states, as in Daniel D. Emmett's song, written in 1859.

Step 3: Starting at the right of the lattice add diagonally, carrying as necessary.



Step 4: Read the answer around the left side and bottom:

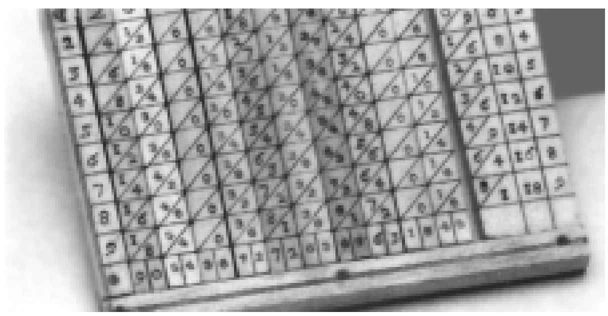
$$38 \times 794 = 30{,}172.$$



John Napier's most significant mathematical contribution, developed over a period of at least 20 years, was the concept of *logarithms,* which, among other things, allow multiplication and division to be accomplished with addition and subtraction. It was a great computational advantage given the state of mathematics at the time (1614).

Napier himself regarded his interest in mathematics as a recreation, his main involvements being political and religious. A supporter of John Knox and James I, he published a widely read anti-Catholic work which analyzed the Biblical book of Revelation. He concluded that the Pope was the Antichrist and that the Creator would end the world between 1688 and 1700. Napier was one of many who, over the years, have miscalculated the end of the world.

The Scottish mathematician John Napier (1550–1617) introduced a significant calculating tool called **Napier's rods,** or **Napier's bones.** Napier's invention, based on the lattice method of multiplication, is widely acknowledged as a very early forerunner of modern computers. It consisted of a set of strips, several for each digit 0 through 9, on which multiples of each digit appeared in a sort of lattice column. See Figure 2.



Index	0	1	2	3	4	5	6	7	8	9
1	0/0	0/1	0/2	0/3	0/4	0/5	0/6	0/7	0/8	0/9
2	0/0	0/2	0/4	0/6	0/8	1/0	1/2	1/4	1/6	1/8
3	0/0	0/3	0/6	0/9	1/2	1/5	1/8	2/1	2/4	2/7
4	0/0	0/4	0/8	1/2	1/6	2/0	2/4	2/8	3/2	3/6
5	0/0	0/5	1/0	1/5	2/0	2/5	3/0	3/5	4/0	4/5
6	0/0	0/6	1/2	1/8	2/4	3/0	3/6	4/2	4/8	5/4
7	0/0	0/7	1/4	2/1	2/8	3/5	4/2	4/9	5/6	6/3
8	0/0	0/8	1/6	2/4	3/2	4/0	4/8	5/6	6/4	7/2
9	0/0	0/9	1/8	2/7	3/6	4/5	5/4	6/3	7/2	8/1

FIGURE 2

An additional strip, called the index, could be laid beside any of the others to indicate the multiplier at each level. Napier's rods were used for mechanically multiplying, dividing, and taking square roots. Figure 3 shows how to multiply 2806 by 7. Select the rods for 2, 8, 0, and 6, placing them side by side. Then using the index, locate the level for a multiplier of 7. The resulting lattice, shown at the bottom of the figure, gives the product 19,642.

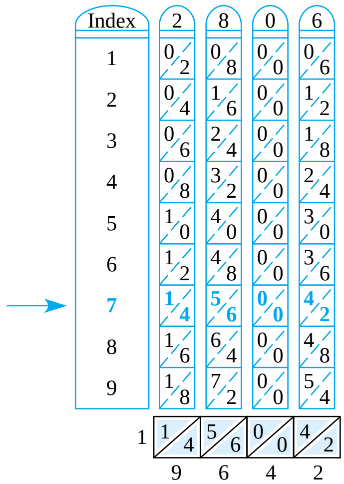


FIGURE 3

EXAMPLE 10 Use Napier's rods to find the product of 723 and 4198.

We line up the rods for 4, 1, 9, and 8 next to the index as in Figure 4. The product 3 × 4198 is found as described in Example 9 and written at the bottom of the figure. Then 2 × 4198 is found similarly and written below, shifted one place to the left. (Why?) Finally, the product 7 × 4198 is written shifted two places to the left. The final answer is found by addition to obtain

$$723 \times 4198 = 3{,}035{,}154.$$

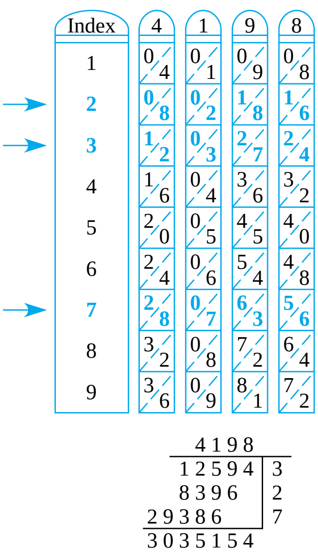


FIGURE 4

One other paper-and-pencil method of multiplication is the **Russian peasant method,** which is similar to the Egyptian method of doubling explained in the previous section. (In fact both of these methods work, in effect, by expanding one of the numbers to be multiplied, but in base two rather than in base ten. Base two numerals are discussed in the next section.) To multiply 37 and 42 by the Russian peasant method, make two columns headed by 37 and 42. Form the first column by dividing 37 by 2 again and again, ignoring any remainders. Stop when 1 is obtained. Form the second column by doubling each number down the column.

	37	42	
	18	84	
Divide by 2, ignoring remainders.	9	168	Double each number.
	4	336	
	2	672	
	1	1344	

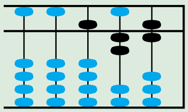
Now add up only the second column numbers that correspond to odd numbers in the first column. Omit those corresponding to even numbers in the first column.

→	**37**	**42**	←
	18	84	
Odd numbers →	**9**	**168**	← Add these numbers.
	4	336	
	2	672	
→	**1**	**1344**	←

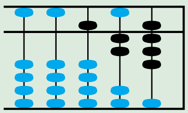
Finally, $37 \times 42 = 42 + 168 + 1344 = 1554$.

FOR FURTHER THOUGHT

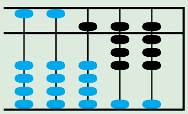
The abacus has been (and still is) used to perform very rapid calculations. A simple example is adding 526 and 362. Start with 526 on the abacus:



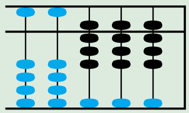
To add 362, start by "activating" an additional 2 on the 1s rod:



Next, activate an additional 6 on the 10s rod:



Finally, activate an additional 3 on the 100s rod:



The sum, read from the abacus, is 888.

For problems where carrying or borrowing is required, it takes a little more thought and skill. Try to obtain an actual abacus (or, otherwise, make sketches) and practice some addition and subtraction problems until you can do them quickly.

For Group Discussion

1. Use an abacus to add: 13,728 + 61,455. Explain each step of your procedure.
2. Use an abacus to subtract: 6512 − 4816. Again, explain each step of your procedure.