4.4 Finite Mathematical Systems

We continue our study of mathematical systems by considering examples built upon *finite* sets. Some examples here will consist of numbers, but others will be made up of elements denoted by letters. And the operations are often represented by abstract symbols with no particular mathematical meaning. This is to make the point that a system is characterized by how its elements behave under its operations, not by the choice of symbols used.

TABLE 10

☆	*a*	*b*	*c*	*d*
a	*a*	*b*	*c*	*d*
b	*b*	*d*	*a*	*c*
c	*c*	*a*	*d*	*b*
d	*d*	*c*	*b*	*a*

To begin, let us introduce a finite mathematical system made up of the set of elements $\{a, b, c, d\}$, and an operation we shall write with the symbol ☆. We give meaning to operation ☆ by displaying an **operation table,** which shows how operation ☆ combines any two elements from the set $\{a, b, c, d\}$. The operation table for ☆ is shown in Table 10. To use the table to find, say, $c \star d$, first locate c on the left, and d across the top. This row and column give b, so that

$$c \star d = b.$$

The important properties to look for in a system are the following: **closure, commutative, associative, identity,** and **inverse.** Let us decide which properties are satisfied by the system made up of $\{a, b, c, d\}$ and operation ☆.

Closure Property

For this system to be closed under the operation ☆, the answer to any possible combination of elements from the system must be in the set $\{a, b, c, d\}$. A glance at Table 10 shows that the answers in the body of the table are all elements of this set. This means that the system is closed. If an element other than a, b, c, or d had appeared in the body of the table, the system would not have been closed.

Commutative Property

In order for the system to have the commutative property, it must be true that $\Gamma \star \Delta = \Delta \star \Gamma$, where Γ and Δ stand for any elements from the set $\{a, b, c, d\}$. For example,

$$c \star d = b \quad \text{and} \quad d \star c = b, \quad \text{so} \quad c \star d = d \star c.$$

TABLE 11

☆	*a*	*b*	*c*	*d*
a	*a*	*b*	*c*	*d*
b	*b*	*d*	*a*	*c*
c	*c*	*a*	*d*	*b*
d	*d*	*c*	*b*	*a*

To see that the same is true for *all* choices of Γ and Δ, observe that Table 11 is symmetric with respect to the diagonal line shown. This "diagonal line test" establishes that ☆ is a commutative operation for this system.

Associative Property

The system is associative if $(\Gamma \star \Delta) \star \Upsilon = \Gamma \star (\Delta \star \Upsilon)$, where Γ, Δ, and Υ represent any elements from the set $\{a, b, c, d\}$. There is no quick way to check a table for the associative property, as there is for the commutative property. All we can do is try some examples. Using the table that defines operation ☆,

$$(a \star d) \star b = d \star b = c, \quad \text{and} \quad a \star (d \star b) = a \star c = c,$$

so that $$(a \star d) \star b = a \star (d \star b).$$

In the same way, $$b \star (c \star d) = (b \star c) \star d.$$

In both these examples, changing the location of parentheses did not change the answers. Since the two examples worked, we suspect that the system is associative. We cannot be sure of this, however, unless every possible choice of three letters from the set is checked. (Although we have not completely verified it here, this system does, in fact, satisfy the associative property.)

Bernard Bolzano (1781–1848) was an early exponent of rigor and precision in mathematics. Many early results in such areas as calculus were produced by the masters in the field; these masters knew what they were doing and produced accurate results. However, their sloppy arguments caused trouble in the hands of the less gifted. The work of Bolzano and others helped put mathematics on a strong footing.

Identity Property For the identity property to hold, there must be an element Δ from the set of the system such that $\Delta \star X = X$ and $X \star \Delta = X$, where X represents any element from the set $\{a, b, c, d\}$. We can see that a is such an element as follows. In Table 11, the column below a (at the top) is identical to the column at the left, and the row across from a (at the left) is identical to the row at the top. Therefore, a is in fact the identity element of the system. (It is shown in more advanced courses that if a system has an identity element, it has *only* one.)

Inverse Property We found above that a is the identity element for the system using operation $\star$. Is there any inverse in this system for, say, the element b? If Δ represents the inverse of b in this system, then

$$b \star \Delta = a \quad \text{and} \quad \Delta \star b = a \quad \text{(since } a \text{ is the identity element).}$$

Inspecting the table for operation $\star$ shows that Δ should be replaced with c:

$$b \star c = a \quad \text{and} \quad c \star b = a.$$

We can inspect the table to see if every element of our system has an inverse in the system. We see (in Table 11) that the identity element a appears exactly once in each row, and that, in each case, the pair of elements that produces a also produces it in the opposite order. Therefore, we conclude that the system satisfies the inverse property.

In summary, the mathematical system made up of the set $\{a, b, c, d\}$ and operation $\star$ satisfies the closure, commutative, associative, identity, and inverse properties.

The basic properties that may (or may not) be satisfied by a mathematical system involving a single operation follow.

Potential Properties of a Single-Operation System

Here a, b, and c represent elements from the set of the system, and $\circ$ represents the operation of the system.

Closure The system is closed if for all elements a and b,

$$a \circ b$$

is in the set of the system.

Commutative The system has the commutative property if

$$a \circ b = b \circ a$$

for all elements a and b from the system.

Associative The system has the associative property if

$$(a \circ b) \circ c = a \circ (b \circ c)$$

for every choice of three elements a, b, and c of the system.

Identity The system has an identity element e (where e is in the set of the system) if

$$a \circ e = a \quad \text{and} \quad e \circ a = a$$

for every element a in the system.

Inverse The system satisfies the inverse property if, for every element a of the system, there is an element x in the system such that

$$a \circ x = e \quad \text{and} \quad x \circ a = e,$$

where e is the identity element of the system.

$\otimes$	**0**	**1**	**2**	**3**	**4**	**5**
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

EXAMPLE 1 The table in the margin is a table for the set $\{0, 1, 2, 3, 4, 5\}$ under an operation designated $\otimes$. Which of the properties above are satisfied by this system?

All the numbers in the body of the table come from the set $\{0, 1, 2, 3, 4, 5\}$, so the system is closed. If we draw a line from upper left to lower right, we could fold the table along this line and have the corresponding elements match; the system has the commutative property.

To check for the associative property, try some examples:

$$2 \otimes (3 \otimes 5) = 2 \otimes 3 = 0 \quad \text{and} \quad (2 \otimes 3) \otimes 5 = 0 \otimes 5 = 0\,,$$

so that

$$2 \otimes (3 \otimes 5) = (2 \otimes 3) \otimes 5\,.$$

Also,

$$5 \otimes (4 \otimes 2) = (5 \otimes 4) \otimes 2\,.$$

Any other examples that we might try would also work. The system has the associative property.

Since the column at the left of the multiplication table is repeated under 1 in the body of the table, 1 is a candidate for the identity element in the system. To be sure that 1 is indeed the identity element here, check that the row corresponding to 1 at the left is identical with the row at the top of the table.

To find inverse elements, look for the identity element, 1, in the rows of the table. The identity element appears in the second row, $1 \otimes 1 = 1$; and in the bottom row, $5 \otimes 5 = 1$; so 1 and 5 both are their own inverses. There is no identity element in the rows opposite the numbers 0, 2, 3, and 4, so none of these elements has an inverse.

In summary, the system made up of the set $\{0, 1, 2, 3, 4, 5\}$ under this operation $\otimes$ satisfies the closure, associative, commutative, and identity properties, but not the inverse property. ■

$\boxtimes$	**1**	**2**	**3**	**4**	**5**	**6**
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

EXAMPLE 2 The table in the margin is a table for the set of numbers $\{1, 2, 3, 4, 5, 6\}$ under an operation designated $\boxtimes$. Which of the properties are satisfied by this system?

Notice here that 0 is not an element of this system. This is perfectly legitimate. Since we are defining the system, we can include (or exclude) whatever we wish. Check that the system satisfies the closure, commutative, associative, and identity properties, with identity element 1. Let us now check for inverses. The element 1 is its own inverse, since $1 \boxtimes 1 = 1$. In row 2, the identity element 1 appears under the number 4, so $2 \boxtimes 4 = 1$ (and $4 \boxtimes 2 = 1$), with 2 and 4 inverses of each other. Also, 3 and 5 are inverses of each other, and 6 is its own inverse. Since each number in the set of the system has an inverse, the system satisfies the inverse property. ■

When a mathematical system has two operations, rather than just one, we can look for an additional, very important property, namely the **distributive property.** For example, when we studied Hindu-Arabic arithmetic of counting numbers, we saw that multiplication is distributive with respect to (or "over") addition.

$$3 \times (5 + 9) = 3 \times 5 + 3 \times 9$$

and

$$(5 + 9) \times 3 = 5 \times 3 + 9 \times 3\,.$$

In each case, the factor 3 is "distributed" over the 5 and the 9.

Although the distributive property (as well as the other properties discussed here) will be applied to the real numbers in general in a later section, we state it here just for the system of integers.

Distributive Property of Multiplication with Respect to Addition

For integers a, b, and c, the **distributive property** holds for multiplication with respect to addition, so

$$\boldsymbol{a \times (b + c) = a \times b + a \times c}$$

and

$$\boldsymbol{(b + c) \times a = b \times a + c \times a.}$$

EXAMPLE 3 Is addition distributive over multiplication?

To find out, exchange $\times$ and $+$ in the first equation above:

$$a + (b \times c) = (a + b) \times (a + c).$$

We need to find out whether this statement is true for *every* choice of three numbers that we might make. Try an example. If $a = 3$, $b = 4$, and $c = 5$,

$$a + (b \times c) = 3 + (4 \times 5) = 3 + 20 = 23,$$

while $$(a + b) \times (a + c) = (3 + 4) \times (3 + 5) = 7 \times 8 = 56.$$

Since $23 \neq 56$, we have $3 + (4 \times 5) \neq (3 + 4) \times (3 + 5)$. This false result is a *counterexample* (an example showing that a general statement is false). This counterexample shows that addition is *not* distributive over multiplication. ■

Because subtraction of real numbers is defined in terms of addition, the distributive property of multiplication also holds with respect to subtraction.

Distributive Property of Multiplication with Respect to Subtraction

For integers a, b, and c, the **distributive property** holds for multiplication with respect to subtraction, so

$$\boldsymbol{a \times (b - c) = a \times b - a \times c}$$

and

$$\boldsymbol{(b - c) \times a = b \times a - c \times a.}$$

The **general form of the distributive property** appears below.

General Form of the Distributive Property

Let $☆$ and $\circ$ be two operations defined for elements in the same set. Then $☆$ is distributive over $\circ$ if

$$\boldsymbol{a ☆ (b \circ c) = (a ☆ b) \circ (a ☆ c)}$$

for every choice of elements a, b, and c from the set.

The final example illustrates how the distributive property may hold for a finite system.

EXAMPLE 4 Suppose that the set $\{a, b, c, d, e\}$ has two operations $\star$ and $\circ$ defined by the tables below.

$\star$	**a**	**b**	**c**	**d**	**e**
a	a	a	a	a	a
b	a	b	c	d	e
c	a	c	e	b	d
d	a	d	b	e	c
e	a	e	d	c	b

$\circ$	**a**	**b**	**c**	**d**	**e**
a	a	b	c	d	e
b	b	c	d	e	a
c	c	d	e	a	b
d	d	e	a	b	c
e	e	a	b	c	d

The distributive property of $\star$ with respect to $\circ$ holds in this system. Verify for the following case: $e \star (d \circ b) = (e \star d) \circ (e \star b)$.

First evaluate the left side of the equation by using the tables.

$$e \star (d \circ b) = e \star e \qquad \text{Use the } \circ \text{ table.}$$
$$= b \qquad \text{Use the } \star \text{ table.}$$

Now, evaluate the right side of the equation.

$$(e \star d) \circ (e \star b) = c \circ e \qquad \text{Use the } \star \text{ table twice.}$$
$$= b \qquad \text{Use the } \circ \text{ table.}$$

Both times the final result is b, and the distributive property is verified for this case. ■