How to do 1.4.16

Theorem 1. *A map $S \to S$.*

(a) *has a left inverse if and only if it is one-to-one,*
(b) *has a right inverse if and only if it is onto.*

Proof: (a) Let $\alpha : S \to S$. Suppose α has a left inverse β. Then $\beta \circ \alpha = \iota_S$, which is one-to-one, hence α is one-to-one by Theorem 2.1.(d).

Conversely, suppose that α is one-to-one. We will construct a left inverse $\beta : S \to S$ as follows. Let $y \in S$. If $y \in \text{Im}(\alpha)$ then there exists $x \in S$ such that $\alpha(x) = y$, so let $\beta(y) = x$. That α is one-to-one tells us that such a preimage is unique. If $y \notin \text{Im}(\alpha)$, then pick any element $s \in S$ and let $\beta(y) = s$.

We need to show that the β we have just constructed is indeed a left inverse to α, that is $\beta \circ \alpha = \iota_S$, that is $\beta(\alpha(x)) = x$ for all $x \in S$. So let $x \in S$. Note that $\alpha(x) \in \text{Im}(\alpha)$, so $\beta(\alpha(x))$ is a preimage of $\alpha(x)$ w.r.t. α. But by one-to-oneness there is exactly element is S that α maps to $\alpha(x)$, namely x. So $\beta(\alpha(x))$ must be equal to x. This is what we wanted to show.

(b) Suppose $\alpha : S \to S$ has a right inverse β. Then $\alpha \circ \beta = \iota_S$, which is onto, hence α is onto by Theorem 2.1.(b).

Conversely, suppose that α is onto. We will construct a right inverse $\beta : S \to S$ as follows. Let $y \in S$. Since α is onto, there exists some $x \in S$ such that $\alpha(x) = y$. There may be more than one such x, but never mind, choose one and let $\beta(y) = x$.

We need to show that the β we have just constructed is indeed a right inverse to α, that is $\alpha \circ \beta = \iota_S$, that is $\alpha(\beta(y)) = y$ for all $y \in S$. So let y be any element of S. By construction, $\beta(y)$ is an element $x \in S$ such that $\alpha(x) = y$. Hence $\alpha(\beta(y)) = y$. □