Math 3124 Exam 2 Solutions

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1. (a) (3 pts) Let G be a group. Define what it means for H to be a subgroup of G.

   We say H is a *subgroup* of G if H is a subset of G and is a group with respect to the operation of G.

   (b) (12 pts) Let G be a group and S a nonempty subset of G. Let

   $$C(S) = \{x \in G \mid xs = sx \ \forall\, s \in S\}.$$

   Prove that $C(S)$ is a subgroup of G. (Warning: S need not be finite.)

   $C(S) \subseteq G$ by definition. Note that $es = se$ for all $s \in S$, so $e \in C(S)$.
   Now let $x, y \in C(S)$. Let $s \in S$. Since $xs = sx$ and $ys = sy$,

   $$(xy)s = x(ys) = x(sy) = (xs)y = (sx)y = s(xy).$$

   This can be done for all $s \in S$, so xy commutes with all $s \in S$. Hence $xy \in C(S)$.
   Let $x \in C(S)$. Multiply $xs = sx$ by x^{-1} on both sides to conclude

   $$\begin{aligned} xs &= sx \\ x^{-1}\, xs\, x^{-1} &= x^{-1}\, sx\, x^{-1} \\ sx^{-1} &= x^{-1}s. \end{aligned}$$

   Doing this for all $s \in S$ shows that $x^{-1} \in C(S)$. Hence $C(S)$ is a subgroup of G.

2. (a) (3 pts) Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^{\geq 0}$ be a distance function. Define what it means for the map $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ to be an isometry.

   A map $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is an *isometry* if it is invertible and

   $$d(\alpha(x), \alpha(y)) = d(x, y) \qquad \forall\, x, y \in \mathbb{R}^2.$$

   (b) (12 pts) Prove that the set of M isometries of the real plane forms a group under composition. You may use the fact that composition of maps is associative. (Hint: for α to be an isometry, it needs to do <u>two</u> things.)

   First let $\alpha, \beta \in M$. Since α and β are both invertible, so they are both one-to-one and onto, hence their composition is also one-to-one and onto by Thm 2.1, and then $\alpha \circ \beta : \mathbb{R}^2 \to \mathbb{R}^2$ is invertible. If $x, y \in \mathbb{R}^2$,

   $$\begin{aligned} d(\alpha \circ \beta(x), \alpha \circ \beta(y)) &= d(\alpha(\beta(x)), \alpha(\beta(y))) \\ &= d(\beta(x), \beta(y)) \\ &= d(x, y) \end{aligned}$$

   where the second equality is because α preserves distances and the third because β preserves distances. So $\alpha \circ \beta \in M$ and $\circ$ is an operation on M. Composition of maps is associative in general.
   Note that $\iota_{\mathbb{R}^2}$ is clearly invertible and preserves distances, so it is in M and works as an identity with respect to composition.

Let $\alpha \in M$. Since α is invertible, $\alpha^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ exists and is also invertible by Exercise 2.20. If $x, y \in \mathbb{R}^2$, use the distance-preserving property of α to see

$$\begin{aligned} d(\alpha^{-1}(x), \alpha^{-1}(y)) &= d(\alpha(\alpha^{-1}(x)), \alpha(\alpha^{-1}(y))) \\ &= d(x, y). \end{aligned}$$

So α^{-1} is also an isometry and $\alpha^{-1} \in M$. Now we can conclude M is a group under composition.

3. (5 pts each)
   (a) Write $(1\ 2\ 3)(1\ 4\ 6\ 5) \in S_6$ in two-row notation.

   $$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 1 & 6 & 2 & 5 \end{pmatrix}$$

   (b) Write $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 3 & 7 & 2 & 1 & 4 \end{pmatrix}$ as a product of disjoint cycles.

   $(1\ 5\ 2\ 6)(3)(4\ 7) = (1\ 5\ 2\ 6)(4\ 7)$

   (c) Write $(2\ 6)(1\ 5\ 2\ 4\ 3\ 6)(5\ 1)$ as a product of disjoint cycles.

   $(1\ 6)(2\ 4\ 3)(5) = (1\ 6)(2\ 4\ 3)$

   (d) Write $(4\ 2\ 5\ 3)$ as a product of transpositions. Is this permutation even or odd?

   $(4\ 3)(4\ 5)(4\ 2)$, which is odd.

4. (5 pts each) **Extra credit problems.**
   (a) Let G be a group in which $x^2 = e$ for all $x \in G$. Prove that G is abelian.

   Let $x, y \in G$. Then $e = (xy)^2 = xyxy$. Multiply on the left by x and on the right by y to get

   $$xey = x(xyxy)y = x^2\, yx\, y^2 = yx.$$

   So $xy = yx$ for any $x, y \in G$, hence G is abelian.

   (b) Let $*$ be an associative operation on the nonempty set S. Suppose that

   $$x * x * y = y = y * x * x \qquad \forall\, x, y \in S.$$

   Prove that S is a group under $*$. Is this group abelian?

   Fix $x \in S$ and let y run through all the elements of S. Since $(x * x) * y = y = y * (x * x)$ for all $y \in S$, $x * x$ acts as an identity. So S has an identity e. In fact, for any $x \in S$, $x * x = e$, so x is its own inverse. We already know $*$ is an operation on S and it is associative, so S is a group under $*$.
   Notice that $x^2 = e$ for any $x \in S$, so by the previous problem, S is abelian.