Name: ______________________ Signature: ____________________

Math 3124 Final Exam

May 3, 2004

The exam is due at **5 PM on Monday, May 10** in the envelope on my office door. You may use your textbook (Durbin), your notes, returned homework, and any solutions to the homework and exams posted on the class website. You are not allowed to use any other books, or the internet, or communicate with anyone else about this exam.

All of your answers must be carefully justified. Neat work, clear and to-the-point explanations will receive more credit than messy, chaotic answers. This exam has 2 pages.

Submit this sheet as the cover sheet of your exam.

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1. (5 pts each) Let G be a group. For an element $x \in G$ define the map $\sigma_x : G \to G$ by

$$\sigma_x(g) = xgx^{-1}.$$

   (a) Prove that σ_x is an automorphism of G.
   (b) Let $\alpha : G \to \mathrm{Aut}(G)$ be the map $\alpha(g) = \sigma_g$. Prove that α is a homomorphism.
   (c) Prove that $\mathrm{im}(\alpha)$ is a normal subgroup of $\mathrm{Aut}(G)$.
   (d) Recall the definition of the center of G from Exercise 7.24:

$$Z(G) = \{g \in G \mid gh = hg \;\forall\, h \in G\}.$$

   Prove that $Z(G)$ is a normal subgroup of G and

$$G/Z(G) \cong \mathrm{im}(\alpha).$$

2. Let G be a group. Define the map $\sigma_x : G \to G$ as in the previous problem. Define the following operation on the set $G \times G$:

$$(x,\, y) * (s,\, t) = (x\sigma_y(s),\, yt).$$

   (a) (10 pts) Prove that $G \times G$ is a group with respect to this operation.
   (b) (5 pts) Let $e \in G$ be the identity and

$$H = \{(g,\, e) \mid g \in G\}.$$

   Prove that H is a normal subgroup of $G \times G$.

3. (5 pts each) Let G be a group. Define the map $\sigma_x : G \to G$ as in Problem 1.
   (a) Let $g \in G$ be fixed. For $x, y \in G$, define the relation $\sim_g$ as $x \sim_g y$ iff $\sigma_x(g) = \sigma_y(g)$. Show that $\sim_g$ is an equivalence relation on G.
   (b) Recall the definition of the centralizer of $g \in G$ from Exercise 7.23:

$$C(g) = \{h \in G \mid hg = gh\}.$$

   We already know this is a subgroup of G, so we can look at its left cosets. Prove that the left cosets of $C(g)$ are the equivalence classes of $\sim_g$.

4. (5 pts each) Let $\sigma, \phi \in S_n$.
   (a) Suppose $\sigma = (a_1\ a_2 \ldots a_k)$. Prove that

$$\phi \circ \sigma \circ \phi^{-1} = (\phi(a_1)\ \phi(a_2) \ldots \phi(a_k)).$$

   (b) Now suppose $\sigma = (a_1\ a_2 \ldots a_k)\,(b_1\ b_2 \ldots b_m)$. Prove that

$$\phi \circ \sigma \circ \phi^{-1} = (\phi(a_1)\ \phi(a_2) \ldots \phi(a_k))\,(\phi(b_1)\ \phi(b_2) \ldots \phi(b_m)).$$

(c) Recall that the cycle structure of σ is defined as the partition $n = n_1 + n_2 + \cdots n_j$ whose parts are the lengths of the disjoint cycles of σ. Conclude from the above that the cycle structure of $\phi \circ \sigma \circ \phi^{-1}$ is the same as the cycle structure of σ. In other words the cycle structure is invariant under conjugation in S_n.

5. (5 pts each) Let R be a ring. Define the center of R as

$$Z(R) = \{x \in R \mid xr = rx \; \forall r \in R\}.$$

(a) Prove that $Z(R)$ is a subring of R.
(b) Find the center of $M_2(\mathbb{Z})$, the ring of 2×2 matrices with integer entries. (Hint: Let E_{ij} be a matrix whose (i,j)-entry is 1, and all other entries are 0. Try multiplying a generic matrix by such matrices on both left and right.)

6. (10 pts) Let $\mathbb{Z}_n$ be the ring as in Example 24.2. Prove that $\overline{k} \in \mathbb{Z}_n$ is a zero divisor if and only if $\gcd(k,n) \neq 1$. Conclude that $\mathbb{Z}_p$ is an integral domain.

7. (15 pts) Let $k_1, k_2, \ldots, k_n \in \mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$. Define the *greatest common divisor* of $k_1, k_2, \ldots, k_n$ as the number $d \in \mathbb{Z}^+$ such that
   1. $d|k_i$ for all $1 \le i \le n$,
   2. if $c|k_i$ for all $1 \le i \le n$, then $c|d$.

   Denote this by $\gcd(k_1, k_2, \ldots, k_n)$.

   Prove that such a number exists, is unique, and is an integer linear combination of $k_1, k_2, \ldots, k_n$. (Hint: you don't have to do this from scratch, you may use what you know about the greatest common divisor of two integers.)

8. (5 pts each) Let $\alpha : G \to H$ be a group homomorphism with $K = \ker(\alpha)$. Let N be a normal subgroup of G such that $N \subseteq K$. Define $\theta : G/N \to H$ as

$$\theta(Ng) = \alpha(g).$$

(a) Prove that θ is well-defined.
(b) Let $\pi : G \to G/N$ be the canonical projection ($\pi(g) = Ng$). Show that $\alpha = \theta \circ \pi$. When this happens, we say α factors through G/N.

9. (5 pts each) **Extra credit problem.** Let R be a ring.
(a) Prove that $(a+b)(a-b) = a^2 - b^2$ for all $a, b \in R$ iff R is commutative.
(b) Prove that $(a+b)^2 = a^2 + 2ab + b^2$ for all $a, b \in R$ iff R is commutative.

10. (5 pts each) **Extra credit problem.** Let $m, n \in \mathbb{Z}$ be relatively prime.
(a) Let and $a, b \in \mathbb{Z}$. Show that there exists $x \in \mathbb{Z}$ such that

$$x \equiv a \mod m$$
$$x \equiv b \mod n.$$

(b) Now suppose that $0 \le a < m$ and $0 \le b < n$. Show that there exists a unique $0 \le x < mn$ such that

$$x \equiv a \mod m$$
$$x \equiv b \mod n.$$

11. (5 pts each) **Extra credit problem.** Let R be a ring with at least two elements. Suppose that for any nonzero $r \in R$ there exists a unique $s \in R$ such that $rsr = r$. Prove the following
(a) R has no zero divisors.
(b) $srs = s$.
(c) R has a multiplicative identity.
(d) Every nonzero element of R has a multiplicative inverse, that is R is a division ring.