Exercise 6.3.15 worksheet

(a) Notation: let c be the capacity map in N and c' be the capacity map in N'.

(i) What is the definition of a cut in the network N?

(ii) What is the definition of a cut in the network N'?

(iii) Prove that (S, T) is a cut in N if and only if it is a cut in N'.

(iv) Let (P, Q) be a cut. What is the definition of $c(P, Q)$? What is the definition of $c'(P, Q)$? What is the relation between $c(P, Q)$ and $c'(P, Q)$?

(v) If (S, T) is a minimal cut in N and (P, Q) is any other cut in N, what can you say about $c(S, T)$ and $c(P, Q)$? What does this tell you about $c'(S, T)$ and $c'(P, Q)$? What do you need to know about d to say so?

(vi) Conclude that if (S, T) is a minimal cut in N then it is a minimal cut in N'.

(vii) If (S, T) is a minimal cut in N' and (P, Q) is any other cut in N', what can you say about $c'(S, T)$ and $c'(P, Q)$? What does this tell you about $c(S, T)$ and $c(P, Q)$? What do you need to know about d to say so?

(viii) Conclude that if (S, T) is a minimal cut in N' then it is a minimal cut in N.

(ix) Do you see much difference in what you had to do in (v)-(vi) and (vii)-(viii)? Can you combine the two arguments into one?

(x) Put the pieces together from (iii), (iv), and (ix) and write up a streamlined proof that (S, T) is a minimal cut in N if and only if it is a minimal cut in N'.

(c) (i) What is the definition of a flow f on N?

(ii) What is the definition of a flow f' on N'?

(iii) Prove that if f is a flow on N then $f' = df$ is a flow on N'. What property of d do you need to use in this proof?

(iv) Prove that if f' is a flow on N' then $f = f'/d$ is a flow on N. What property of d do you need to use in this proof?

(v) Combine the proofs in (iii) and (iv) into one to show that f is a flow on N if and only if f' is a flow on N'.

(vi) Given that $f' = df$, what is the relation between $|f'|$ and $|f|$?

(vii) If f is a maximal flow on N, and g is any other flow on N, what can you say about $|f|$ and $|g|$?

(viii) Suppose f is a maximal flow on N and g' is any flow on N'. Let $f' = df$. Is $g = g'/d$ a flow on N? How do $|f|$ and $|g|$ compare? How do $|f'|$ and $|g'|$ compare?

(ix) Conclude that if f is a maximal flow on N then f' is a maximal flow on N'.

(x) Notice that the argument is symmetric. Let $M = N'$, $d' = 1/d$, and $M' = d'M$. Do M, M', and d' satisfy the conditions of the exercise? If so, apply your conclusion in (ix) to M, M', and d'. Notice that you just proved that if f' is a maximal flow on N', then $f = f'/d$ is a maximal flow on N.

(xi) If you don't like the way we used the symmetry between N and N' to prove the converse, can you do it by turning the argument in (viii) around?

(xii) Distill the wisdom you gained in (v)-(ix) and either (x) or (xi) into a slick proof that f is a maximal flow on N if and only if $f' = df$ is a maximal flow on N'.

(b) (i) What can you say about the values of two different maximal flows on the same network?

(ii) Let f be a maximal flow on N. What did part (c) say about df? What is the relation between $|df|$ and $|f|$?

(iii) Let f' be a maximal flow on N' (f' need not be equal to df here). What is the relation between $|f'|$ and $|df|$? What about between $|f'|$ and $|f|$?

(iv) Combine your arguments in (i)-(iii) into one short proof that if v is the value of a (any) maximal flow on N and v' is the value of a (any) maximal flow on N', then $v' = dv$.