Math 3134 Exam 1 Solutions

Feb 18, 2005

0. (2 pts) Spell correctly the singular of "vertices."
   Vertex.

1. (10 pts) Let G_1 and G_2 be directed or undirected simple graphs and f be an isomorphism from G_1 to G_2. Let E_i be the set of edges of G_i. For edges a and b, say that b *follows* a if a, b is a path. Show that there exists a one-to-one correspondence $g : E_1 \to E_2$ such that for $a, b \in E_1$, $g(b)$ follows $g(a)$ if and only if b follows a. (Note the converse is false.)

   For an edge e denote the starting vertex by $T(e)$ and the ending vertex by $H(e)$ (T and H stand for the tail and head of an arrow).

   We will first define a map $g : E_1 \to E_2$ as follows. Let e be an edge in G_1. Since f is an isomorphism, we know there is an edge e' in G_2 from $f(T(e))$ to $f(H(e))$. Let $g(e) = e'$. Observe that $T(g(e)) = T(e') = f(T(e))$ and $H(g(e)) = H(e') = H(T(e))$.

   Now, we will define a map $h : E_2 \to E_1$. Let e' be an edge in G_2. Since f^{-1} is an isomorphism from G_2 to G_1 (remember the proof of 3.1.28), there is an edge e'' from $f^{-1}(T(e'))$ to $f^{-1}(H(e'))$. Let $h(e') = e''$.

   If $e \in E_1$, then $g(e)$ is an edge from $f(T(e))$ to $f(H(e))$ in G_2, and $h(g(e))$ is an edge from $f^{-1}(f(T(e))) = T(e)$ to $f^{-1}(f(H(e))) = H(e)$ in G_1. But there is exactly one edge from $T(e)$ to $H(e)$ and that is e. So $h(g(e)) = e$ for all $e \in E_1$. By a similar argument, $g(h(e')) = e'$ for all $e' \in E_2$. Hence h is the inverse of g, and then g must be a one-to-one correspondence.

   Let $a, b \in E_1$. Notice that b follows a iff $T(b) = H(a)$. Since f is one-to-one, $T(b) = H(a)$ iff $f(T(b)) = f(H(a))$. This happens iff $T(g(b)) = H(g(a))$ iff $g(b)$ follows $g(a)$.

   Remarks:
   - Many of you like to claim in your homework that an isomorphism f from G_1 to G_2 is a one-to-one correspondence of the vertices and the edges. This is false because by definition f is a map $V_1 \to V_2$ and it does not map $E_1 \to E_2$. But if you have f, you can construct a one-to-one correspondence $g : E_1 \to E_2$ by the above argument.
   - You have already done part of this exercise when you showed that a graph isomorphism sends a path to a path in the homework assigned on 2/2. If you did that problem right, you constructed g and showed that $g(b)$ follows $g(a)$ if b follows a, although you most likely did not put it in these same terms.
   - Why is the converse false?

2. (a) (3 pts) Let f be a map from the set S to the set T. Define what it means for f to be onto.

   f is onto if $f(S) = T$. Alternately, f is onto if for all $t \in T$ there is an $s \in S$ such that $f(s) = t$.

   (b) (3 pts) Let f be a map from the set S to the set T. Define what it means for f to be one-to-one.

   f is one-to-one if for all $x, y \in S$ such that $x \neq y$, $f(x) \neq f(y)$. Alternately, f is one-to-one if for all $x, y \in S$, $f(x) = f(y)$ implies $x = y$.

Remark: Saying that f assigns to every element of S a unique element of T is merely the definition of a map and does not characterize one-to-one maps.

(c) (10 pts) Let $f : A \to B$ and $g : B \to C$ be one-to-one correspondences. Prove that $g \circ f$ is a one-to-one correspondence.

So f and g are both one-to-one and onto, and we need to show the same for $g \circ f$.
Let $x, y \in A$ such that $x \neq y$. Then $f(x) \neq f(y)$ because f is one-to-one and $g(f(x)) \neq g(f(y))$ because g is one-to-one. So $g \circ f$ is one-to-one.
As f is onto, $f(A) = B$. Since g is onto, $g(f(A)) = g(B) = C$, which shows that $g \circ f$ is onto.

(d) (7 pts) Let S be a set of sets and $\sim$ the following relation on S. For $A, B \in S$, $A \sim B$ iff there is a one-to-one correspondence $f : A \to B$. Prove that $\sim$ is an equivalence relation. (You may use the theorem from the book which states a one-to-one correspondence has an inverse, and the inverse is also a one-to-one correspondence.)

Let $A, B, C \in S$.
Reflexivity: Since the identity map $i_A : A \to A$ is obviously one-to-one and onto (it's its own inverse), $A \sim A$.
Symmetry: If $A \sim B$ there exists a one-to-one correspondence $f : A \to B$. By Theorem 2.9, $f^{-1} : B \to A$ exists and is a one-to-one correspondence. Hence $B \sim A$.
Transitivity: If $A \sim B$ and $B \sim C$, then there exist one-to-one correspondences $f : A \to B$ and $g : B \to C$. By part (c), $g \circ f : A \to C$ is a one-to-one correspondence. So $A \sim C$.

3. (a) (5 pts) Let G be an undirected graph. State a theorem that gives a necessary and sufficient condition for the existence of an Eulerian circuit in G in terms of the degrees of the vertices. You don't need to prove the theorem.

G has an Eulerian circuit if and only if G is connected up to isolated vertices and the degree of each vertex is even.

(b) (10 pts) Let G be an undirected graph. Recall that the line graph $L(G)$ of G is the undirected graph whose vertices correspond to the edges of G, and two vertices are adjacent if and only if the corresponding edges in G share an endpoint.
Prove that G has an Eulerian circuit if and only if $L(G)$ has a Hamiltonian cycle. Conclude that if G has an Eulerian circuit then $L(G)$ is connected.

Say two edges e_1 and e_2 are neighboring if they share an endpoint. Let v_1 and v_2 be the corresponding vertices in $L(G)$. By definition of $L(G)$, e_1 and e_2 are neighboring iff v_1 and v_2 are adjacent.
Suppose G has an Eulerian circuit $P = e_1, e_2, \ldots, e_n$. Let v_i be the vertex which corresponds to e_i in $L(G)$. Since P is a path, e_i and e_{i+1} are neighboring, so v_i is adjacent to v_{i+1}. Since P is a circuit, e_n and e_1 are neighboring, so v_n is adjacent to v_1. Hence $\overline{P} = v_1, v_2, \ldots, v_n, v_1$ is a circuit. As P contains each edge of G exactly once, $\overline{P}$ contains each vertex of $L(G)$ exactly once, hence it is a Hamiltonian cycle.
The proof of the converse is very similar. If $L(G)$ has a Hamiltonian cycle $\overline{P} = v_1, v_2, \ldots, v_n, v_1$, let e_i be the edge in G which corresponds to v_i. The adjacency

of the consecutive vertices in $\overline{P}$ implies that consecutive edges in the sequence $P = e_1, e_2, \ldots, e_n$ are neighboring and so are e_n and e_i. Hence P is a circuit. Since $\overline{P}$ contains each vertex of $L(G)$ exactly once, so P contains each edge of G exactly once, hence it is an Eulerian circuit.
If G has an Eulerian circuit, then $L(G)$ has a Hamiltonian cycle. We can get from any vertex to any other by traveling along this cycle, hence $L(G)$ is connected.

Remark: In the line graph problem on the homework, you had to show that if G has an Eulerian circuit then $L(G)$ does too. This proof involves showing that $L(G)$ is connected. If you did this right, you must have come up with a very similar argument to the above, in which you converted a path in G to a path in $L(G)$ just like here.