Math 3134 Exam 3 Solutions

Apr 15, 2005

1. (15 pts) Let G be a finite, undirected graph.

   (a) Give the definition of a matching of G.

   A matching M is a subset of the edges of G such that no two edges in M are incident to the same vertex.

   (b) Let M be a matching and C a covering of G. Prove that $|M| \leq |C|$ and if $|M| = |C|$, then M is a maximal matching and C is a minimal covering.

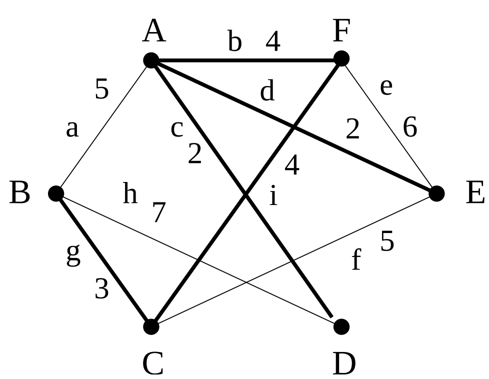
   We will construct a map $\alpha : M \to C$. Let $e \in M$. Since C is a covering, there exists a vertex $V \in C$ which is incident to M. There may be more than one such vertices, but we can always choose one. Now, let $e, e' \in M$ such that $e \neq e'$. Since M is a matching, $\alpha(e) \neq \alpha(e')$. So α is one-to-one. Hence $|M| = |\alpha(M)| \leq |C|$.
   Suppose $|M| = |C|$. If M is not maximal, then there exists a matching M' with $|M'| > |M|$. But then $|M'| > |C|$ which contradicts the previous statement. If C is not minimal, then there exists a covering C' with $|C'| < |C|$. But then $|C'| < |M|$ which contradicts the previous statement.

2. (20 pts)

   (a) Give Prim's algorithm for finding a minimal spanning tree in a finite, undirected, weighted graph. You don't have to prove that the algorithm works.

   1. Let V be a random vertex in G, and set $P = \{V\}$ and $E = \emptyset$.
   2. While there is an edge from a vertex in P to a vertex not in P
      2.1. Among all edges with one end in P and the other outside P choose e with minimal weight
      2.2. Let V be the endpoint of e that is not in P, and $E := E \cup \{e\}$, $P := P \cup \{V\}$.
      end while
   3. If $P = V_G$ then return (P, E),
      otherwise print "There is no minimal spanning tree because G is unconnected."
      end if

   (b) Run the algorithm on the graph below starting at vertex C to find a minimal spanning tree. Make sure to indicate the order in which the algorithm selects the edges.



$\{g\}$
$\{g, i\}$
$\{g, i, b\}$
$\{g, i, b, d\}$
$\{g, i, b, d, c\}$

Note: In the last two steps, c and d could be selected in either order.

3. (5 pts) In a small Appalachian town live the taxpayers Amy-Lu, Billy-Bob, Cletus, Dolly-Sue, and Earleen. As Apr 15 approaches, they want to minimize their tax liability by claiming dependents. There are six children in the town called Jeb, Kellie-Mae, Lester, Mary-Jo, Ollie, and Peggy. Everyone's last name is Johnson. According to IRS rules, no child can be claimed as a dependent by more than one adult. To be on the safe side, no one will risk claiming a child that is not related to them. Amy-Lu is related to Jeb, Lester, and Peggy. Billy-Bob is related to Jeb, Kellie-Mae, Mary-Jo, and Ollie. Cletus is related to Jeb and Peggy. Dolly-Sue is related to Lester and Peggy. Earleen is related to Jeb, Lester, and Peggy. Can the adults divide up the children among themselves so that everyone can claim at least one?

Here are the five sets corresponding to the five adults:

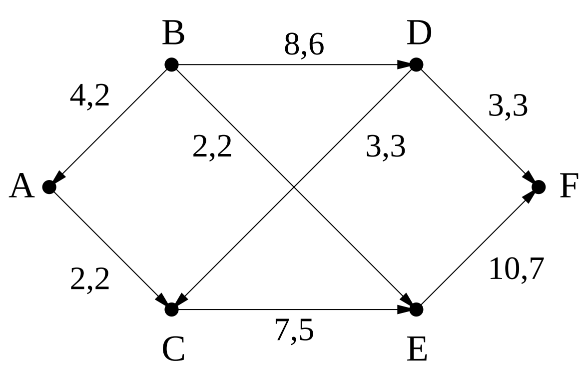
$$A = \{J, L, P\}$$
$$B = \{J, K, M, O\}$$
$$C = \{J, P\}$$
$$D = \{L, P\}$$
$$E = \{J, L, P\}.$$

Notice that $|A \cup C \cup D \cup E| = |\{J, L, P\}| = 3$, so by Hall's Theorem, there exists no system of distinct representatives, which is exactly what the problem asked for.

4. (10 pts)
(a) Let N be a network with capacity c. Give the definition of a flow f on N.

A flow is a map $f : N \to \mathbb{R}^{\geq 0}$ such that
1. $f(e) \leq c(e)$ for all edges $e \in E_N$,
2. for any vertex V of N other than the source and the sink, the sum of the flows on the incoming edges is equal to the sum of the flows on the outgoing edges.

(b) Find a flow of value 10 for the network below.
(The numbers on the edges are the capacities followed by the flow.)



The easy way to find such a flow is to notice that the cut $(\{A, B, D\}, \{C, E, F\})$ has capacity 10. Hence the edges (A, C), (B, E), (D, C), (D, F) must be used at full capacity. The remaining flows now follow easily.

5. (10 pts) **Extra credit problem.** Let G be a finite, undirected, connected, weighted graph. Prove that if all the weights of G are different then G has exactly one minimal spanning tree. (Hint: If there are two minimal spanning trees, look for an edge of smallest weight among those that are in exactly one of these two trees.)

Since G is connected it has a spanning tree and therefore a minimal spanning tree.

Suppose there are more than one. Let T_1 and T_2 be different minimal spanning trees. Among all edges that are in T_1 but not T_2 or in T_2 but not T_1, let e be the one with minimal weight. Without loss of generality, e is in T_1. Just like in the proof of Prim's algorithm, add e to T_2 to get T_2'. T_2' now has a cycle and this cycle has at least one edge e' which is not in T_1 because T_1 does not have the same cycle. Remove e' from the resulting cycle to get a spanning tree T_2''. Obviously, $e' \neq e$. By the choice of e, $w(e) < w(e')$, so $w(T_2'') < w(T_2)$. This contradicts the fact that T_2 was a minimal spanning tree.

Note: Showing that if the weights are all distinct then Prim's algorithm always picks the same minimal spanning tree regardless of which vertex it starts does not prove the statement. Prim's algorithm finds <u>a</u> minimal spanning tree of a connected graph, or finds <u>some</u> minimal spanning trees if started at different vertices, but <u>not all</u> in general. There could still exist another minimal spanning tree that Prim's algorithm just doesn't find.