Math 313 Exam 1 Solutions

Oct 4, 2006

1. (20 pts)

   (a) Define what a prime number is. Give an example of a number that is prime and explain why it is. Give an example of a number that is not prime and explain why it is not.

   A prime number is an integer $p \geq 2$ which is only divisible by 1 and itself.

   Alternately: A prime number is a positive integer which has exactly two divisors.

   Examples:

   - 5 is a prime number because it is only divisible by 1 and 5.
   - 6 is not a prime number because in addition to 1 and 6, it is divisible by 2.

   (b) State the Fundamental Theorem of Arithmetic. Give an example of a prime factorization.

   A positive integer $n \geq 2$ can be written as a product of primes. This product is unique except for the order in which the prime factors are listed.
   E.g. $20 = 2^2 \cdot 5$.

   (c) Use Eratosthenes's sieve to find the prime numbers up to 30. Explain what you are doing.

   Eratosthenes's sieve works as follows:

   (i) Write down the integers from 2 to 30. Let $n = 2$.

   (ii) Cross out all multiples of n except for n itself.

   (iii) If there is a number greater than n which is not already crossed out, then set n equal to this number and repeat (ii).

   When done, the numbers that are crossed out are the composite numbers, and the numbers that are left are the primes.

	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~

   (d) Explain why Eratosthenes's sieve eliminates the composite numbers but leaves the prime numbers.

   First notice that the sieve can only cross out composite numbers because it only eliminates numbers that are multiples of a smaller number that is different from 1. So it never crosses out a prime. On the other hand, a number that is composite is guaranteed to be crossed out at the same time that any one of its prime divisors is circled. So the sieve crosses out every composite number and leaves every prime.

2. (10 pts) Find a 7-digit number which is divisible by 4 and 5, but not by 3 or 8. Explain in detail how you got your answer and why your method should give a correct answer.

That the number must divisible by 5 tells us that its last digit is 0 or 5. That it must be divisible by 4 tells us that the number formed by its last two digits is divisible by 4. But it cannot be divisible by 8, so the number formed by the last three digits must not be divisible by 8. Applying these three requirements, we choose the last three digits so they are not divisible by 8 but are divisible by 4 and 5, for example 020. Now we just have to choose the remaining digits in such a way that the sum of the digits is not divisible by 3. One such choice is 1100020.

3. (10 pts) How many factors does the number $5^4 \cdot 6^3 \cdot 7^2$ have? Explain in detail how you got your answer and why your method should give the correct answer.

We can easily determine the number of factors of a number by looking at its prime factorization. So first we find the prime factorization. Notice that 5 and 7 are primes, but $6 = 2 \cdot 3$. Hence $6^3 = 2^3 \cdot 3^3$ and

$$5^4 \cdot 6^3 \cdot 7^2 = 2^3 \cdot 3^3 \cdot 5^4 \cdot 7^2$$

Clearly, a factor of this number cannot have a prime factor other than $2, 3, 5$, or 7. Hence any factor has prime factorization $2^i \cdot 3^j \cdot 5^k \cdot 7^l$. This number must be an integer, so we know $0 \leq i, j, k, l$. Also

$$\frac{2^3 \cdot 3^3 \cdot 5^4 \cdot 7^2}{2^i \cdot 3^j \cdot 5^k \cdot 7^l} = 2^{3-i} \cdot 3^{3-j} \cdot 5^{4-k} \cdot 7^{2-l}$$

must be integer, hence

$$\begin{aligned} 0 \geq 3 - i &\iff i \leq 3 \\ 0 \geq 3 - j &\iff j \leq 3 \\ 0 \geq 4 - k &\iff k \leq 3 \\ 0 \geq 2 - l &\iff l \leq 3 \end{aligned}$$

We can choose i to be 0,1,2, or 3. For each of these four choices, we can choose j to be 0,1,2, or 3. For each such choice, we have 5 choices for k (0,1,2,3, or 4) and for each of those, we have 3 choices for l (0,1, or 2). This will give us $4 \cdot 4 \cdot 5 \cdot 3 = 240$ choices. Each of these corresponds to a different factor. Therefore we have 240 factors.

4. (10 pts) Find $\operatorname{lcm}(3^2 \cdot 11^4,\ 2^5 \cdot 3^3,\ 2 \cdot 5^2 \cdot 11)$. Explain how you get your answer and why your method should give the correct answer.

Let $a = 3^2 \cdot 11^4$, $b = 2^5 \cdot 3^3$, and $c = 2 \cdot 5^2 \cdot 11$. To find the least common multiple from the prime factorizations, choose the largest power of each prime that shows up and multiply these powers together. This gives $n = 2^5 \cdot 3^3 \cdot 5^2 \cdot 11^4$. Since

$$\begin{aligned} \frac{n}{a} &= 2^5 \cdot 3 \cdot 5^2 \\ \frac{n}{b} &= 5^2 \cdot 11^4 \\ \frac{n}{c} &= 2^3 \cdot 3^3 \cdot 11^3 \end{aligned}$$

are all integer, we see that n is indeed a multiple of a, b, and c.

But if we tried removing a 2 or a 3 from the prime factorization of n, then it would no longer be a multiple of b. If we removed a 5, it wouldn't be a multiple of c, and if we removed an 11, it wouldn't be a multiple of a. Hence n is indeed the least among all common multiples of a, b, and c.

5. (15 pts) **Extra credit problem.** This is a harder problem. Attempt it only when you are done with everything else.

The divisibility tests we have learned depend on the fact that we write our numbers to base 10. If you expressed the same number in a different numeral system, the sum of the digits and the last digits would be different.

The Nenets people, who live in northern Russia, use a nonary system, which means they write their numbers to base 9.

(a) If you were to open a Nenets math textbook, you would find the following divisibility test for 3: a number is divisible by 3 if and only if the last digit is 0 or 3 or 6. Explain why this is true. (Hint: you may start by checking on a few examples that the rule really works.)

Let x be a number that is written as $d_n d_{n-1} \dots d_1 d_0$ in the nonary system (where the $d_0, \dots, d_n$ are digits and $0 \le d_i \le 8$ for all i). This means

$$x = d_n 9^n + d_{n-1} 9^{n-1} + \dots + d_1 9 + d_0$$

Since $9^n, 9^{n-1}, \dots, 9$ are all divisible by 3, the divisibility of x by 3 is determined by d_0. In fact, x is divisible by 3 exactly when d_0 is, which happens when d_0 is 0,3, or 6.

(b) In Nenets math, the divisibility test for 2 says that a number is divisible by 2 if and only if the sum of the digits of the number is divisible by 2. Explain why this is true. (Hint: play with a few examples first.)

Let's use the same notation as in part (a). Notice that all the powers of 9 are odd because they are products of odd numbers. Now if d_i is odd, then it $d_i 9^i$ is odd, otherwise $d_i 9^i$ is even. Since the sum of any number of even numbers is even, whether x is even is decided by how many of the $d_i 9^i$ are odd, which in turn is decided by how many of the d_i are odd. x is going be even exactly when an even number of the d_i are odd.
Now, $d_n + d_{n-1} + \dots + d_0$ is also even exactly when an even number of d_i are odd. So we may conclude that x is divisible by 2 exactly when the sum of the digits is divisible by 2.

Here is an example. Let $x = 325_9$. In our decimal system,

$$x = 3 \cdot 9^2 + 2 \cdot 9 + 5 = \underbrace{3 \cdot 81}_{\text{odd}} + \underbrace{2 \cdot 9}_{\text{even}} + \underbrace{5}_{\text{odd}} = 266$$

(c) Find a divisibility test for 4 that works in the Nenets numeral system. Show on an example that your test indeed works. Explain why your test works in general.

Let's use the same notation as in part (a). Notice that $1, 9, 81, 729$ and in fact any power of 9 has a remainder of 1 if try to divide it by 4 using integer division. That is $9^i - 1$ is always divisible by 4. Now notice that

$$d_i 9^i = d_i 9^i \underbrace{-d_i + d_i}_{0} = \underbrace{d_i(9^i - 1)}_{\text{divisible by 4}} + d_i$$

Hence

$$\begin{aligned}
x &= d_n 9^n + d_{n-1} 9^{n-1} + \cdots + d_1 9 + d_0 \\
&= \underbrace{d_n(9^n - 1) + d_n}_{d_n 9^n} + \underbrace{d_{n-1}(9^{n-1} - 1) + d_{n-1}}_{d_{n-1} 9^{n-1}} + \cdots + \underbrace{d_1(9-1) + d_1}_{d_n 9^n} + d_0 \\
&= \underbrace{d_n(9^n - 1) + d_{n-1}(9^{n-1} - 1) + \cdots + d_1(9-1)}_{\text{divisible by 4}} + \underbrace{d_n + d_{n-1} + \cdots d_1 + d_0}_{\text{sum of the digits}}
\end{aligned}$$

So x is divisible by 4 exactly when the sum of its digits is divisible by 4.

Here is an example. Let $x = 147_9$. In our decimal system,

$$\begin{aligned}
x &= 1 \cdot 9^2 + 4 \cdot 9 + 7 = 1 \cdot 81 + 4 \cdot 9 + 7 \\
&= \underbrace{1 \cdot 80 + 1}_{1 \cdot 81} + \underbrace{4 \cdot 8 + 4}_{4 \cdot 9} + 7 = \underbrace{1 \cdot 80 + 4 \cdot 8}_{\text{divisible by 4}} + 1 + 4 + 7
\end{aligned}$$

Since $1 + 4 + 7 = 12$ is divisible by 4, x should be divisible by 4. Indeed $x = 124$, which is divisible by 4.

Notice that this is the same argument as the one that shows that in the decimal system, a number is divisible by 9 (or 3) exactly when the sum of its digits is.