Math 313 Exam 2 Solutions

Nov 8, 2006

1. (20 pts) Consider an isosceles (having two equal sides) right-angle triangle like the one shown further down on the left.

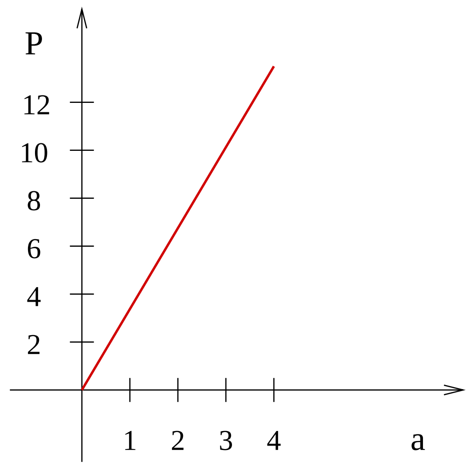
   (a) Draw a graph that shows how the perimeter of the triangle depends on the length of side a. (Hint: Use the Pythagorean theorem to find the length of b. You may or may not find it useful to know that $\sqrt{2} \approx 1.4$.)

   By the Pythagorean Theorem

   $$b^2 = a^2 + a^2$$

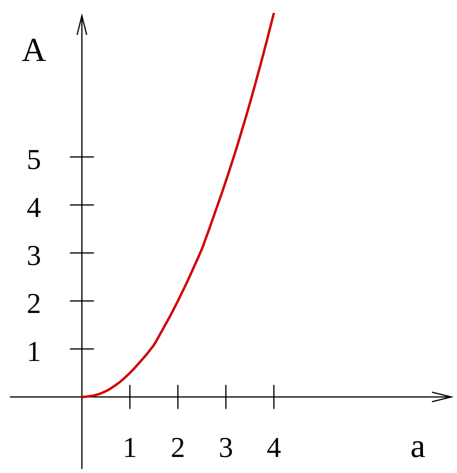
   $$b = \sqrt{2a^2} = \sqrt{2}a$$

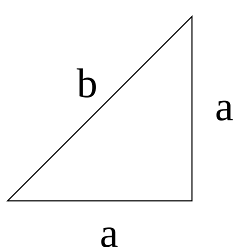
   So $P = 2a + b = (2 + \sqrt{2})a \approx 3.4a$. Hence the graph is a line of slope $2 + \sqrt{2} \approx 3.4$ starting at the origin.



   (b) Draw a graph that shows how the area of the triangle depends on the length of side a.

   Notice that the height that corresponds to side a is also a. So the area of the triangle is $A = \frac{a^2}{2}$. Hence the graph is a parabola:





   (c) What can you say about the slope of the graph in part (b)? Is the graph a line?

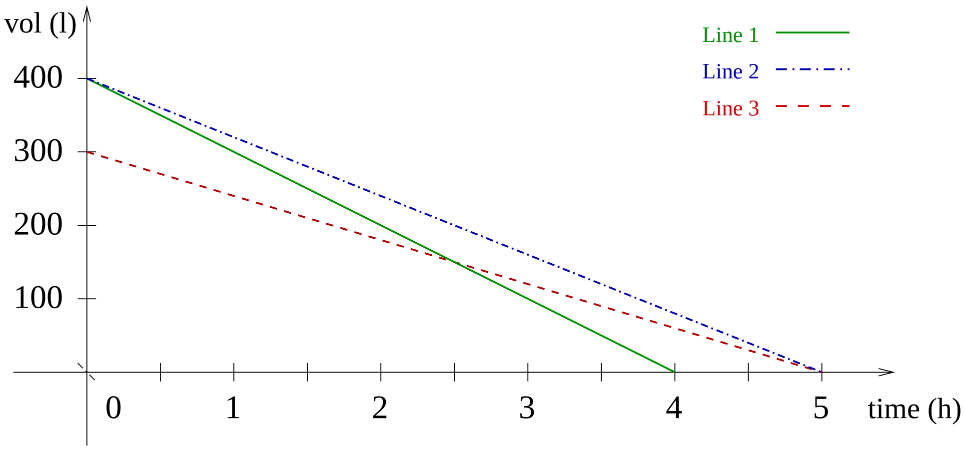
   The graph is not a line because $A = \frac{a^2}{2}$ is a quadratic function of a. Its slope is not constant and we can see that the slope becomes steeper as a increases.

   (d) Are the perimeter of the triangle and its area related? As one increases, how does the other change? Why?

   Yes, they are related. The only way for the perimeter to increase is if a increases because the angles cannot change. Otherwise the triangle would cease to have a right-angle or two equal sides. If a increases, a^2 also increases, and hence the area increases.

Remark: Notice this problem is very similar to problem 12.2.3 on your homework. Only I cut the square in half along one of its diagonals to produce the triangle.

2. (10 pts) At the beer festival (Chevefest) in Mexicali, there are three kinds of beer on tap: Cucapá Trigueña, Cucapá Obscura, and Tijuana Güera. Everyone at the festival is equally thirsty (and drinks at the same rate), but 1/3 of the people prefer Cucapá Trigueña, 1/4 of the people prefer to drink Cucapá Obscura, and the rest drink TJ Güera. The graph below shows the level of the three kinds of beer in the kegs over time.



(a) Find the slopes (including physical units) of the lines in the graph. Then identify which line corresponds to which kind of beer.

The slopes are the following:

$$\text{Line 1} \qquad m = \frac{-400\,\text{l}}{4\,\text{h}} = -100\,\frac{\text{l}}{\text{h}}$$
$$\text{Line 2} \qquad m = \frac{-400\,\text{l}}{5\,\text{h}} = -80\,\frac{\text{l}}{\text{h}}$$
$$\text{Line 3} \qquad m = \frac{-300\,\text{l}}{5\,\text{h}} = -60\,\frac{\text{l}}{\text{h}}$$

If 1/3 of the people drink Cucapá Trigueña and 1/4 drink Cucapá Obscura, then $1 - 1/3 - 1/4 = 5/12$ of the people drink TJ Güera. Notice that the ratios of these fractions are $3 : 4 : 5$, exactly like the ratios of the slopes above. The most popular beer is TJ Güera, so this must decrease the fastest, hence it corresponds to line 1. The second most popular is Cucapá Trigueña, so this must correspond to line 2. Finally, the least popular beer is Cucapá Obscura, so this must decrease the the most slowly, hence it corresponds to line 3.

(b) Which group of three kegs would you say is best represented by the graph above? Why? Identify which keg contains which beer.

Group 1 Group 2 Group 3

Looking at the y-intercepts of the lines, we see that initially we had 400 l of TJ Güera and Cucapá Trigueña and 300 l of Cucapá Obscura. The only group that has two equal size kegs and one smaller is Group 2. In that group, the two large kegs must contain the TJ Güera and Cucapá Trigueña and the small keg must have the Cucapá Obscura.

Remark: This is just like the candle problem (12.3.4) on the HW, isn't it.

3. (30 pts) Speedy Gonzalez sets out to the store for tortillas. Initially, he runs fast (as befits his name), but about halfway to the store reaches a random checkpoint by the border patrol and he has to join a slow-moving line. When he gets up to the checkpoint, he realizes that he he forgot to bring proper documentation with him. He immediately turns around and runs back home at the same speed as before. At home, he spends a few minutes looking for his papers. Mrs. Gonzalez is upset that there are still no tortillas for lunch, so she tells Speedy he'd better hurry. So Speedy sets out again, this time running faster than before. He finds the checkpoint gone, so he can continue his dash all the way to the store. He spends a few minutes in the store, buying the tortillas. With the weight of 5 dozen tortillas, he can't run as fast he had done initially (on his way to the checkpoint), so he returns home at leisurely jog.

(a) Use the three coordinate systems on the next page to draw qualitative graphs of the following: position-time (that is Speedy's distance from home as a function of time), distance-time (that is the total distance Speedy's traveled as a function of time), and speed-time (that is Speedy's speed as a function of time). Label your axes. Make sure that your graphs are consistent with the story and with each other, that is they don't contradict each other.

See graphs on next page.

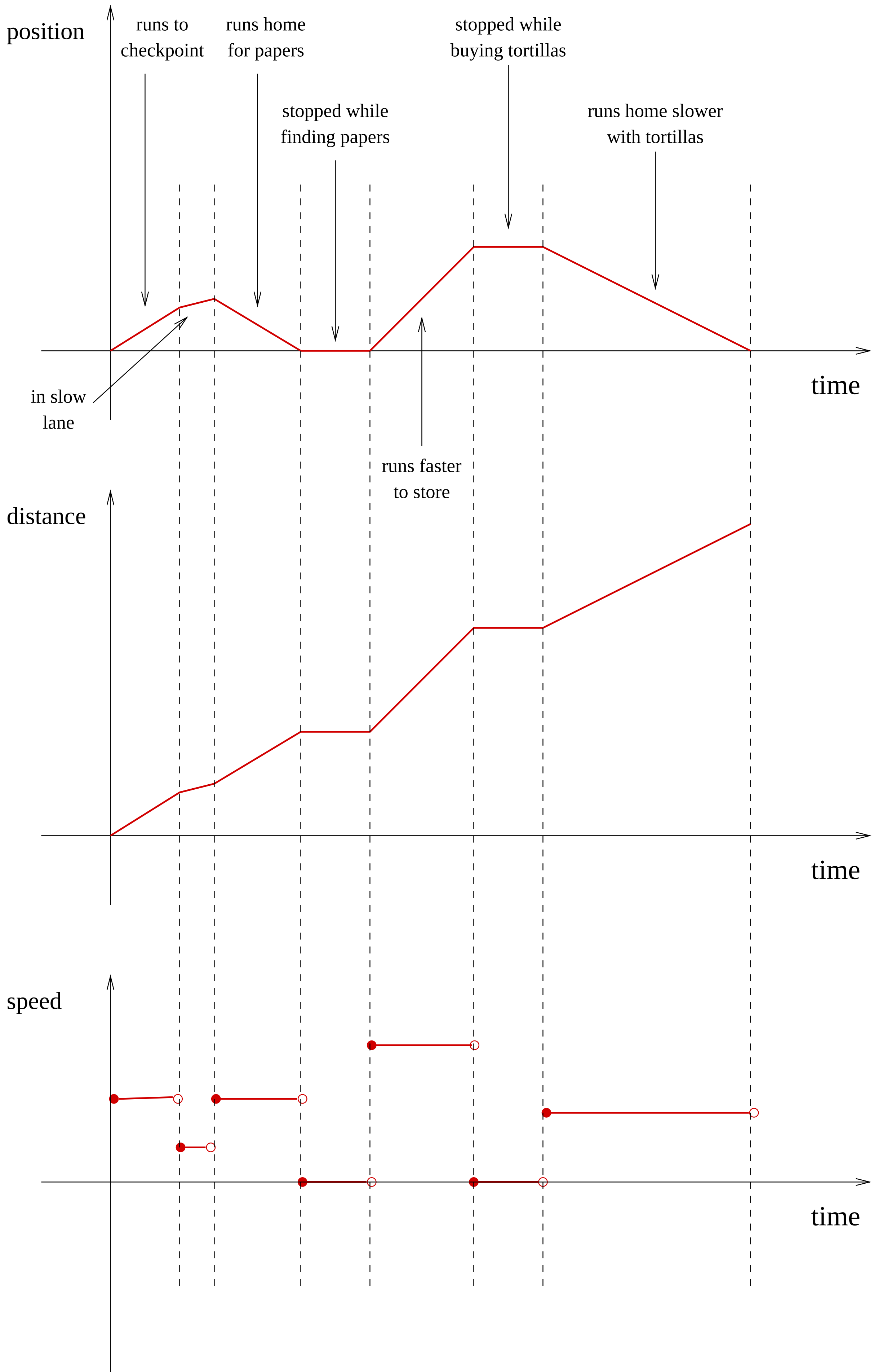
(b) Compare the position-time graph with the distance-time graph. What do they have in common? How exactly do they differ?

The various types of motion take place at the same time on both graphs, so the individual pieces of the graphs must start and end at the same times. Since they depict Speedy Gonzalez moving at the same speed, the magnitudes of the slopes of the pieces must also be the same. The signs of the slopes can be opposite. In particular, distance covered cannot decreases, so its slope is never negative, whereas the slope of the position is negative every time Speedy is running toward his home.

(c) Compare the speed-time graph with either the position-time graph or the distance-time graph, whichever you think is more appropriate. How exactly are they related? How do they differ?

My speed-time graph is more closely related to my distance-time graph. Again, the various types of motion take place at the same time on both graphs, so the individual pieces of the graphs must start and end at the same times. The speed-time graph is the slope of the distance-time graph.
The obvious difference is that the speed-time graph and the distance-time graph measure different physical quantities. Speed is measured in units of distance/units of time (e.g. miles/hour), whereas distance is measured in distance units (e.g. miles).

Remark: Can you tell which exercise(s) from the HW this problem resembles. Is there a pattern here? Am I being predictable?


position
runs to checkpoint
runs home for papers
stopped while buying tortillas
stopped while finding papers
runs home slower with tortillas
in slow lane
time
runs faster to store
distance
time
speed
time

4. (15 pts) **Extra credit problem.** This is a harder problem. Attempt it only when you are done with everything else.

   Find the smallest positive integer that gives a remainder of 1 if divided by 4, a remainder of 2 if divided by 5, and a remainder of 3 if divided by 6. Carefully explain your reasoning.

   Let's call the number we are looking for n. Notice that if n gives a remainder of 1 if divided by 4, then $n + 3$ is divisible by 4. Similarly, $n + 3$ is divisible by 5. In fact, $n + 3$ must be divisible by 6 too. And we are looking for the smallest such positive integer. Then $n + 3$ must be $\text{lcm}(4, 5, 6) = 60$. So $n = 57$.

   Remark: This problem is a Math Olympiads problem from 1983. The Math Olympiads (`www.moems.org`) are a series of competitions that elementary and junior high school students can enter five times a year. (The problem above is for junior high students.)