# Math 313 Final Exam Solutions

Dec 13, 2006

1. A jail in a castle has a thousand cells numbered 1 to 1000 and each cell holds one prisoner. The jailer has the key that operates the lock of each cell door. Turning the key unlocks the door if it was locked, and locks it if it was unlocked. The doors are of course normally kept locked.

   Feeling jolly about Christmas, the king decides on the following amnesty scheme. The jailer is first asked to go through every cell and turn the key once in each lock. Then he is asked to go through every second cell and turn the key once. Then he is asked to go through every third cell and turn the key once. Then every fourth. And so on. On his last round, he turns the key once on the 1000th cell. At the end of this process, every prisoner, whose cell is unlocked is released.

   (a) (5 pts) Is the prisoner is cell #375 released? Why or why not?

   No. According to the general argument given in (c), only prisoners whose cell number is a perfect square will be released. $375 = 3 \cdot 5^3$ is not a square.

   (b) (5 pts) Is the prisoner is cell #144 released? Why or why not?

   Yes. According to the general argument given in (c), only prisoners whose cell number is a perfect square will be released. $144 = 2^4 \cdot 3^2 = 12^2$ is a square.

   (c) (10 pts) Formulate a general statement on which prisoners are released and which are not. Justify your statement.

   The jailer turns the key once for each number that is a divisor of the cell's number. A prisoner will be released exactly when the key is turned an odd number of times in the lock of his cell. That is the prisoner goes free if his cell number has an odd number of divisors. The divisors of a number typically come in pairs. If $x$ divides $y$, so does $y/x$. So we can pair up $x$ and $y/x$. The only exception is when $x = y/x$, which happens when $y = x^2$. So a perfect square has a divisor which cannot be paired up because its pair would be itself. Most positive integers have an even number of divisors, except for squares, which have an odd number of divisors. Therefore a prisoner will be released exactly when his cell number is a square.

2. (10 pts each)

   (a) Do there exist non-zero whole numbers $m$ and $n$ such that $3^m = 9^n$? If yes, find all of the solutions, if not carefully explain why not.

   First notice

   $$9^n = (3^2)^n = 3^{2n}$$

   So we are trying to solve the equation $3^m = 3^{2n}$. If $n > 0$, then $3^{2n} > 1$, so $3^m > 1$ and then $m > 0$ too. In this case, $3^m$ and $3^{2n}$ are integer. The Fundamental Theorem of Arithmetic says that every integer larger than 1 factors into a product of primes uniquely except for the order of the prime factors. Therefore $m$ and $2n$ must be equal. To find all solutions in this case, we would choose any positive integer for $n$ and we would set $m = 2n$.
   If $n < 0$, then $3^{2n} < 1$, so $3^m < 1$ and then $m < 0$ too. To make it easier to see what to do, we will set $a = -m$ and $b = -n$. Now, $a$ and $b$ are positive integers. We need to

solve the equation

$$\begin{aligned} 3^{-a} &= 3^{-2b} \\ \frac{1}{3^a} &= \frac{1}{3^{2b}} && \text{take the reciprocal} \\ 3^a &= 3^{2b} \end{aligned}$$

We are now back in the first case we discussed and we already know how to find all the solutions. We choose any positive integer for $b$ and we would set $a = 2b$. This is the same thing as choosing any negative integer for $n$ and setting $m = 2n$.
To summarize, we can get all the non-zero solutions to $3^m = 9^n$ by choosing any non-zero integer for $n$ and set $m = 2n$.

(b) Do there exist non-zero whole numbers $m$ and $n$ such that $5^m = 7^n$? If yes, find all of the solutions, if not carefully explain why not.

The same idea works here as in part (a). If $m > 0$, then $5^m > 1$, so $7^n > 1$ and $n > 0$. In this case, $5^m$ and $7^n$ are integer. Both $5^m$ and $7^n$ would have to be prime factorizations of the same integer. This would violate the Fundamental Theorem of Arithmetic. Hence there can be no solutions.
If $m < 0$ then $5^m < 1$, so $7^n < 1$ and $n < 0$. We can now do what we did in part (a). Set $a = -m$, $b = -n$, so $a$ and $b$ are positive integers.

$$\begin{aligned} 5^{-a} &= 7^{-b} \\ \frac{1}{5^a} &= \frac{1}{7^b} && \text{take the reciprocal} \\ 5^a &= 7^b \end{aligned}$$

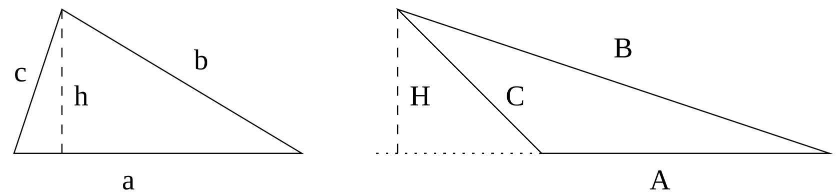
We already know this equation has no positive solutions. So this case doesn't give any solutions for $m$ and $n$ either.
Therefore there exist no non-zero integers $m$ and $n$ such that $5^m = 7^n$.

(Hints for both parts: Keep in mind $m$ and $n$ could be negative too. The Fundamental Theorem of Arithmetic may help with this problem.)

3. (10 pts each) In the following decide if the first quantity is related to the second quantity. If they are related, sketch a graph showing the relationship. You may want to start by constructing a table of values.

(a) Does the area of a triangle depend on its perimeter?

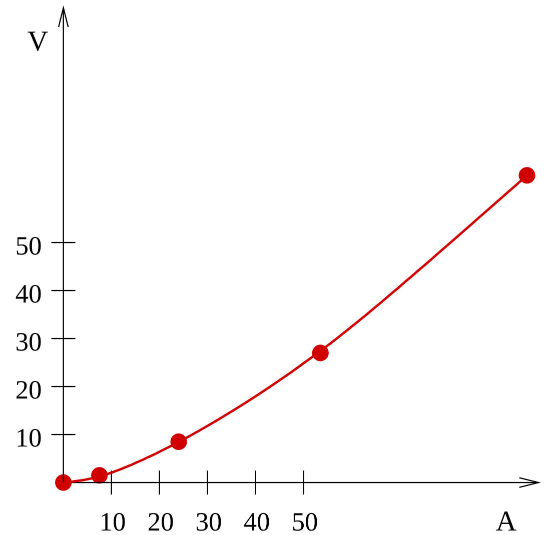
The area of a triangle is not related to its perimeter. We can easily change the perimeter of a triangle without changing the area. The picture below shows two triangles which have the same area because $a = A$ and $h = H$, but different perimeters because $b < B$ and $c < C$.



(b) Does the volume of a cube depend on the surface area of the cube? (Hint: Is each related to the side of the cube?)

Yes. First note that if the side of the cube is $x$, then $V = x^3$ and $A = 6x^2$. The latter is true because the surface of the cube consists of six squares. To increase the surface area of a cube, we need to increase its side, which will increase its volume. Same thing with decrease. See the table of values and the graph below.

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $A$ | 0 | 6 | 24 | 54 | 96 |
| $V$ | 0 | 1 | 8 | 27 | 64 |



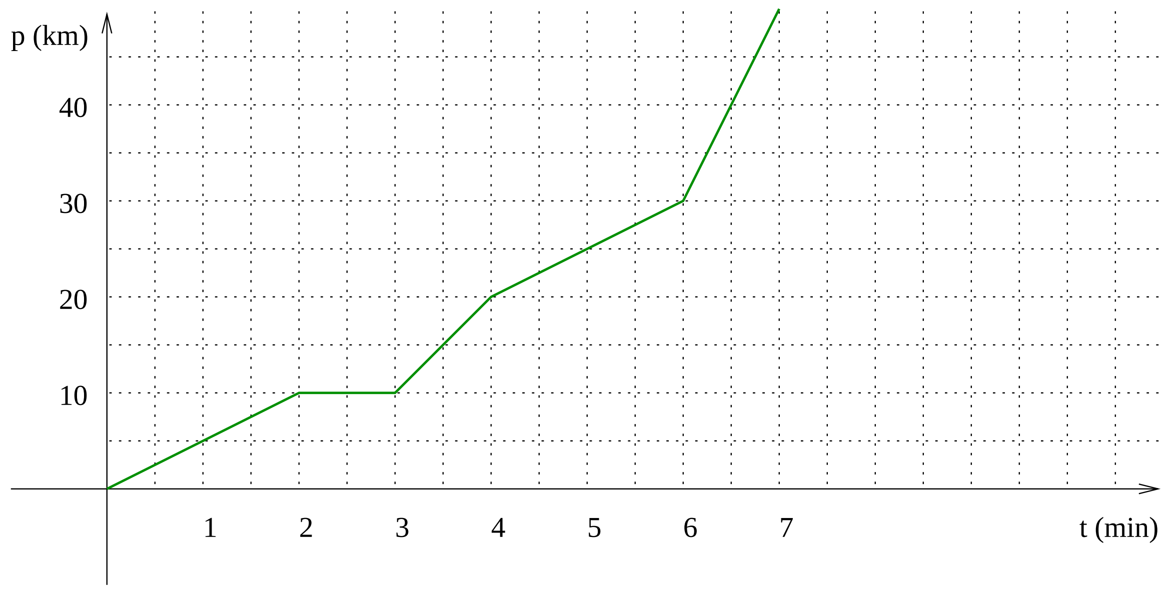

If you want to know the algebraic equation that relates the volume to the surface area, you'll first have to solve for $x$ in terms of $A$, which gives $x = \sqrt{A/6}$. Now you can substitute this into $V = x^3$:

$$V = x^3 = \sqrt{\frac{A}{6}}^{\,3}$$

4. (10 pts each) A position-time graph is given below.

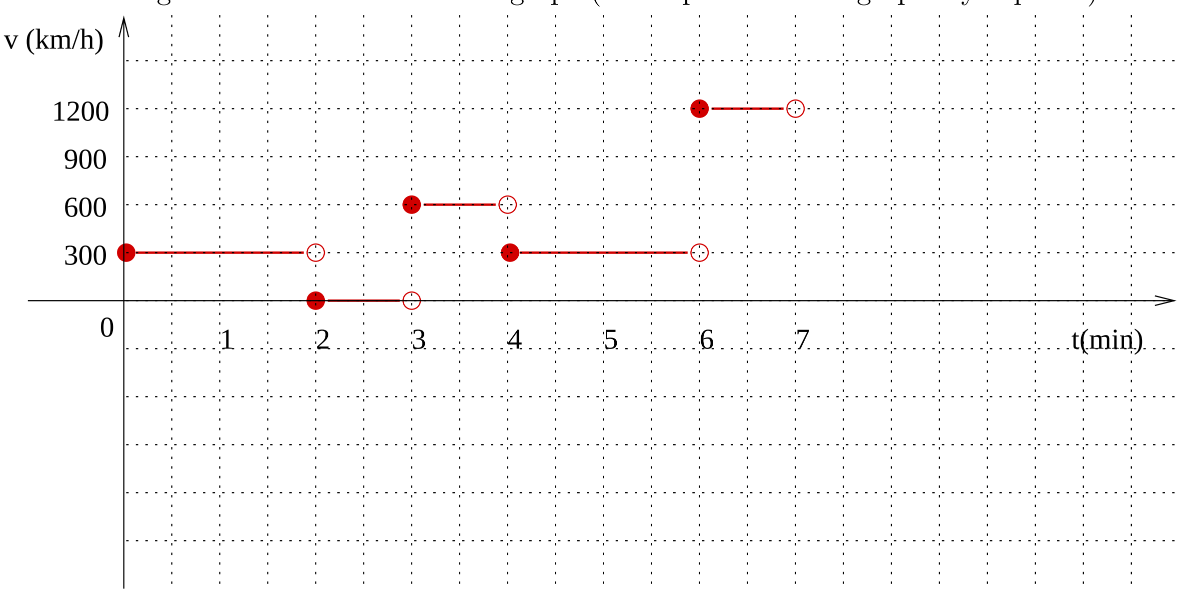

(a) Use the position-time graph to draw a distance-time graph.

Distance always changes at the same rate as position, only distance always increases. So the position-time and the distance-time graph have the same magnitude slopes, only the slope of the distance-time graph is never negative. Also, the distance time-graph has to start at the origin, because at $t = 0$, no distance has been covered yet.




(b) Use the position-time graph to draw a speed-time graph.

Speed is the rate of change of distance (or position, according to our text, although that's usually called velocity). So you can get the speed-time graph by taking the slopes of the line segments in the distance-time graph (or the position-time graph if you prefer).




(c) Come up with a reasonable story that fits the position-time graph above.

Luke Skywalker was patrolling the frosty planet Hoth 10 km from the rebel base, when he felt hungry. So he headed back to the base at a speed of 300 km/h. Once there, it took him only a minute to fix a PBJ sandwich. Then he jumped back in his X-wing and returned to where he'd left off at twice the speed as before. When he got there, he slowed down to 300 km/h and resumed his patrol. He flew another 10 km, when he felt a sudden disturbance in the force. Worried about the safety of his fellow rebels at the base, he pulled a screeching U-turn and floored his X-wing. He returned to the base at 1200 km/h.

Luke's X-wing is equipped with the new No-Acceleration$^{TM}$ unit. The No-Acceleration$^{TM}$ unit enables a spacecraft to change its speed and direction instantaneously without accelerating/decelerating. It uses highly experimental technology and is not yet available to the general public.

5. (10 pts) "Mr. Hunt, we have a very dangerous situation developing in North Korea. We understand that a rogue military faction very hostile to the West is planning to take control of that country's nuclear arsenal and launch a missile against our ally, Japan. This must be prevented. Unfortunately, we don't know which missile and what the target is. If we did, our missile defense system would be able to take it out. Given the highly strained relations between North Korea and the West, we cannot rely on their government for our security. The President has signed orders for a pre-emptive strike to take place at 1800Z against all nuclear launch sites in North Korea. Obviously, this is a very risky and undesirable outcome. Should our intelligence on NK's nuclear weapons prove incomplete, they would be able to launch a counterattack with dire consequences.

   There is just one more thing that can be done. We are in contact with a South Korean double-agent, who has ties to the rogue members of NK's military. He says he can steal the CDs that contain the exact locations and programmed targets of NK's nuclear missiles. These are stored in a secure bunker at a NK military airfield. Your assignment, should you choose to accept it, is to fly a disguised Mig-29 from one of our bases in the Pacific to South Korea, pick up the double-agent, fly him to this North Korean airfield, assist him in stealing the CDs, and return them to our base. Once we have the CDs, we can effectively use our missile defense system to thwart the planned attack. Given the extremely delicate nature of this mission, discretion is utmost. We will not be able to provide any assistance once you take off. You will not be able to communicate with us until you return.

   Your aircraft will be ready for take-off at 1150Z. Your aircraft's cruise airspeed (the speed relative to surrounding air) is 750 mph. The distance from our base to South Korea is 1500 miles. From your landing site there to the airfield in NK is 750 miles, from NK back to the base 1500 miles. On the way to SK, you will be flying at 40000 ft against a 150 mph headwind, giving you a ground speed of 600 mph. The forecast predicts no wind between SK and NK. On your way back from NK to the base, you will be flying with a 150 mph tailwind, giving you a ground speed of 900 mph. According to my calculations, that should give you and average speed of 750 mph and your flight will take 5 hours. You will need an hour to steal the CDs. You have 6 hours and 10 minutes to complete your mission."

   Do you see anything wrong with the general's computation? If you are Ethan Hunt, do you accept the assignment? Why or why not?

   Flying from the base to SK will take

$$\frac{1500\,\text{miles}}{600\,\frac{\text{miles}}{\text{h}}} = 2\frac{1}{2}\,\text{h}$$

From SK to NK, there is no wind, so the Mig-29 will fly at 750 mph and will take exactly 1 hour. From NK back to the base, we will need

$$\frac{1500\,\text{miles}}{900\,\frac{\text{miles}}{\text{h}}} = 1\frac{2}{3}\,\text{h}$$

So the total flight time is actually

$$2\frac{1}{2}\,\text{h} + 1\,\text{h} + 1\frac{2}{3}\,\text{h} = 5\frac{1}{6}\,\text{h}$$

That is 5 hours and 10 min. The general seems to have made the mistake of simply averaging 600 mph, 750 mph, and 900 mph to arrive at

$$\frac{600\,\text{mph} + 750\,\text{mph} + 900\,\text{mph}}{3} = 750\,\text{mph}$$

We know this is wrong. The actual average speed is going to be less because we spend more time flying at 600 mph than at 900 mph. He really should have used a weighted average, or could have computed the average speed by dividing the total distance (3750 miles) by the total flying time (51/6 h).

Including the 1 hour we need in NK to steal the CDs, we need exactly 6 hours and 10 min. If there is no unexpected delay anywhere, we can complete the mission by 1800Z. It's tight, but no problem for an Ethan Hunt kind of guy. Saving the world is worth it, so I'd accept the mission.

6. **Extra credit problem.** This is a harder problem. Attempt it only when you are done with everything else.

   (a) (10 pts) You learned in elementary school that dividing by a fraction $x/y$ is the same thing as multiplying by $y/x$. Explain why this is true.

   There are different possible explanations. You would choose one depending on the mathematical sophistication of your audience. Here is a fairly low-level one. Let's say we are dividing $z$ by $x/y$:

   $$\frac{z}{\frac{x}{y}} = \frac{z}{x\frac{1}{y}} = \frac{z}{x}\frac{1}{\frac{1}{y}} = \frac{z}{x}y = z\,\frac{y}{x}$$

   where $\frac{1}{1/y} = y$ because the number of times you can fit $1/y$ into 1 is exactly $y$.
   Another explanation is that dividing by a number means multiplying by its multiplicative inverse (reciprocal). And the the multiplicative inverse of $x/y$ is in fact $y/x$ because

   $$\frac{x}{y}\frac{y}{x} = \frac{xy}{yx} = 1$$

   (b) (5 pts) You learned in elementary school that you cannot divide any number by 0. Give a convincing explanation why not.

   Suppose $x$ is a non-zero number. If we could divide $x$ by 0, then $x/0$ would have to be some number $y$. If $x/0 = y$, then $0 \cdot y$ would have to be $x$. But $0 \cdot y = 0$, which cannot be $x$ because $x$ was non-zero. If $x$ is actually 0, see the answer to part (c).

   Alternately, you could rephrase $x/0$ as how many times do I have to add 0 to itself to get $x$. No matter how many times you add 0 together, you get 0. So if $x \neq 0$, then no answer makes sense to $x/0$.

   (c) (10 pts) After explaining to your 6th grade class that you cannot divide any number by 0, Lisa Simpson raises her hand and says "We know that 2/2=1, 5/5=1, 3.7/3.7=1. So shouldn't we say 0/0=1?" Explain to Lisa why 0/0 is not 1.

   Suppose $0/0 = 1$. Surely, $0 \cdot 0 = 0$. So

   $$1 = \frac{0}{0} = \frac{0 \cdot 0}{0} = 0\,\frac{0}{0} = 0 \cdot 1 = 0$$

   This is bad. If we believed $0/0 = 1$, we would also have to believe $1 = 0$, which is not true.