Math 315 Exam 1 Solutions

Feb 20, 2015

1. (5 pts each)

   (a) Show that

$$\frac{-1+\sqrt{3}i}{2}$$

   is a cube root of 1 (meaning that its cube is equal to 1).

$$\begin{aligned}\left(\frac{-1+\sqrt{3}i}{2}\right)^3 &= \frac{(-1+\sqrt{3}i)^3}{2^3}\\ &= \frac{(-1)^3+3(-1)^2\sqrt{3}i+3(-1)(\sqrt{3}i)^2+(\sqrt{3}i)^3}{8}\\ &= \frac{-1+3\sqrt{3}i+9+-3\sqrt{3}i}{8}\\ &= 1\end{aligned}$$

   Hence $\frac{-1+\sqrt{3}i}{2}$ is a cube root of 1.

   (b) Find two distinct square roots of i.

   Let $\alpha = x+yi$. For α to be a square root of i, it has to satisfy

$$i=\alpha^2=(x+yi)^2=x^2-y^2+2xyi$$

   Hence $x^2-y^2=0$ and $2xy=1$. First,

$$x^2-y^2=0 \implies x^2=y^2 \implies y=\pm x.$$

   If $y=-x$

$$1=2xy=-2x^2 \implies x^2=-\frac{1}{2}$$

   which has no real solution for x. If $y=x$

$$1=2xy=2x^2 \implies x^2=\frac{1}{2} \implies x=\pm\frac{1}{\sqrt{2}} \implies y=\pm\frac{1}{\sqrt{2}}.$$

   This gives

$$\alpha=\pm\frac{1+i}{\sqrt{2}}$$

   as the two square roots of i.

2. (10 pts) Show that in the definition of a vector space, the additive inverse condition can be replaced with the condition that $0v=0$ for all $v\in V$. Here 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V. (The phrase "a condition can be replaced" in a definition means that the collection of objects satisfying the definition is unchanged if the original condition is replaced with the new condition.) Hint: show that if V is a vector space that satisfies the original definition we gave in class then it also satisfies $0v=0$ for all $v\in V$; and conversely, if V is a vector space that satisfies the new definition, then every $v\in V$ must have an additive inverse.

   Call V an old vector space if it satisfies the original definition of a vector space we gave in a class and a new vector space if it satisfies the same axioms except instead of the existence of additive inverses we have $0v = 0$ for all $v \in V$. We are supposed to show that V is an old vector space if and only if it is a new vector space.

So suppose V is an old vector space. Then Proposition 1.29 shows that V also satisfies $0v = 0$ for all $v \in V$, so V is also a new vector space.

Now suppose V is a new vector space. Let $v \in V$. We will show that v has an additive inverse in V. Let $w = (-1)v$. Since V is closed under scalar multiplication, $w \in V$. In fact, w is an additive inverse to v:

$$\begin{aligned} v + w &= v + (-1)v \\ &= 1v + (-1)v && \text{because } v = 1v \\ &= (1 + (-1))v && \text{by distributivity} \\ &= 0v \\ &= 0 && \text{since } V \text{ is a new vector space} \end{aligned}$$

Similarly, $w + v = 0$. Therefore every $v \in V$ has an additive inverse, and hence V is an old vector space.

3. (5 pts each) Let V be a vector space and v any vector in V.
   (a) Prove that v has a unique additive inverse in V.

   See Proposition 1.26 in your textbook.

   (b) Prove that $(-1)v = -v$.

   See Proposition 1.31 in your textbook.

4. (10 pts) Let $V = \mathbb{R}^\infty$ be the vector space of real valued sequences over the field $\mathbb{R}$. Let $a \in \mathbb{R}$ and

$$U = \{(x_1, x_2, \ldots) \in V \mid \lim_{n\to\infty} x_n = a\}.$$

Show that U is a subspace of V if and only if $a = 0$. (Hint: you may use what you learned about limits of sequences in calculus and/or precalculus.)

This is similar to Example 1.35(a). First, suppose $a \neq 0$. Notice that the 0 sequence $(0, 0, \ldots)$ has limit 0, so if $a \neq 0$, then this sequence is not in U and hence U cannot be a subspace.

Now suppose $a = 0$. Then $(0, 0, \ldots) \in U$, so U is nonempty. Suppose $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots)$ are in U. Then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$. By the usual properties of limits,

$$\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n = 0 + 0 = 0.$$

Hence

$$(x_1, x_2, \ldots) + (y_1, y_2, \ldots) = (x_1 + y_1, x_2 + y_2, \ldots) \in U.$$

Now, let $(x_1, x_2, \ldots) \in U$ and $\alpha \in \mathbb{R}$. Then

$$\lim_{n\to\infty}(\alpha x_n) = \alpha \lim_{n\to\infty} x_n = \alpha 0 = 0.$$

Hence

$$\alpha(x_1, x_2, \ldots) = (\alpha x_1, \alpha x_2, \ldots) \in U.$$

Therefore U is a subspace.

5. (10 pts) **Extra credit problem.** Let V be a vector space and $\{U_\alpha\}_{\alpha \in I}$ a collection of (possibly infinitely many) subspaces of V where I is some indexing set. Define the sum of the U_α as

$$\sum_{\alpha \in I} U_\alpha = \left\{ \sum_{\alpha \in I} u_\alpha \mid u_\alpha \in U_\alpha \ \forall \alpha \in I \text{ and } u_\alpha = 0 \text{ for all but finitely many } \alpha \right\}.$$

Prove that $\sum_{\alpha \in I} U_\alpha$ is the smallest subspace of V which contains all of the U_α. (Hint: follow the proof we gave in class for the finite case and think carefully about what to do differently.)

This can be shown much the same way as Proposition 1.39. First, notice that since $0 \in U_\alpha$ for all $\alpha \in I$,

$$0 = \sum_{\alpha \in I} 0 \in \sum_{\alpha \in I} U_\alpha.$$

Now, suppose $v, w \in U$. Then there exist some finite subset A of I and vectors $v_\alpha \in U_\alpha$ for $\alpha \in A$ such that $v = \sum_{\alpha \in A} v_\alpha$. Similarly, there exist some finite subset B of I and vectors $w_\alpha \in U_\alpha$ for $\alpha \in B$ such that $w = \sum_{\alpha \in B} w_\alpha$. Hence

$$v + w = \sum_{\alpha \in A} v_\alpha + \sum_{\alpha \in B} w_\alpha = \sum_{\alpha \in A \setminus B} v_\alpha + \sum_{\alpha \in B \setminus A} w_\alpha + \sum_{\alpha \in A \cap B} (v_\alpha + w_\alpha) \in \sum_{\alpha \in I} U_\alpha$$

since the sum on the right hand side consists of finitely many elements from the various U_α.

If if $v \in U$ and $\lambda \in F$, then $v = \sum_{\alpha \in A} v_\alpha$ for some finite subset A of I and vectors $v_\alpha \in U_\alpha$. Hence

$$\lambda v = \lambda \sum_{\alpha \in A} v_\alpha = \sum_{\alpha \in A} \underbrace{\lambda v_\alpha}_{\in U_\alpha} \in \sum_{\alpha \in I} U_\alpha.$$

Therefore $\sum_{\alpha \in I} U_\alpha$ is a subspace of V.

Now, we need to show $\sum_{\alpha \in I} U_\alpha$ is the smallest susbspace that contains all of the U_α. Let W be any subspace of V which contains all of the U_α. Let v be any vector in $\sum_{\alpha \in I} U_\alpha$. Then $v = \sum_{\alpha \in A} v_\alpha$ for some finite subset A of I and vectors $v_\alpha \in U_\alpha$. Since $u_\alpha \in W$ for all $\alpha \in A$, and W is closed under addition, $v \in W$. As this is true for any $v \in \sum_{\alpha \in I} U_\alpha$, it must be that $\sum_{\alpha \in I} U_\alpha \subseteq W$.