Math 315 Final Exam Solutions

May 11, 2015

1. (10 pts) Let V be a vector space. Is the operation of addition on the subspaces of V associative? In other words, if U_1, U_2, U_3 are subspaces of V, is
$$(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)?$$
(Hint: Subspaces are sets. Think carefully about what it means to show that two sets are equal.)

   We will prove it is. Let U_1, U_2, U_3 be subspaces of V. To prove $(U_1+U_2)+U_3 = U_1+(U_2+U_3)$, we need to show $(U_1+U_2)+U_3 \subseteq U_1+(U_2+U_3)$ and $U_1+(U_2+U_3) \subseteq (U_1+U_2)+U_3$. Let $v \in (U_1 + U_2) + U_3$. Then $v = (u_1 + u_2) + u_3$ for some $u_1 \in U_1$, $u_2 \in U_2$, and $u_3 \in U_3$. By associativity of addition on V,
$$v = (u_1 + u_2) + u_3 = u_1 + (u_2 + u_3) \in U_1 + (U_2 + U_3).$$
This is true for all $v \in (U_1 + U_2) + U_3$, so $(U_1 + U_2) + U_3 \subseteq U_1 + (U_2 + U_3)$. A symmetric argument shows $U_1 + (U_2 + U_3) \subseteq (U_1 + U_2) + U_3$.

2. (10 pts) Suppose U and W are subspaces of the vector space V such that $V = U \oplus W$ Suppose also that $u_1, \dots, u_m$ is a basis of U and $w_1, \dots, w_n$ is a basis of W. Prove that
$$u_1, \dots, u_m, w_1, \dots, w_n$$
is a basis of V.

   We will first show that any vector in $U+W$ is a linear combination of the $u_1, \dots, u_m, w_1, \dots, w_n$. Let $v \in U + W$. So $v = u + w$ for some $u \in U$ and $w \in W$. Since $u_1, \dots, u_m$ span U and $w_1, \dots, w_n$ span W, we can write
$$u = \alpha_1 u_1 + \cdots + \alpha_m u_m$$
$$w = \beta_1 w_1 + \cdots + \beta_n w_n$$
for some $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n \in F$. Hence
$$v = \alpha_1 u_1 + \cdots + \alpha_m u_m + \beta_1 w_1 + \cdots + \beta_n w_n.$$

   Now, to show that $u_1, \dots, u_m, w_1, \dots, w_n$ is linearly independent, suppose
$$\alpha_1 u_1 + \cdots + \alpha_m u_m + \beta_1 w_1 + \cdots \beta_n w_n = 0.$$
Then the vector
$$v = \alpha_1 u_1 + \cdots + \alpha_m u_m = -\beta_1 w_1 - \cdots - \beta_n w_n$$
is both in U and W. Since $U + W$ is direct, $U \cap W = \{0\}$. So $v = 0$. Now, $\alpha_i = 0$ for all i by the linear independence of $u_1, \dots, u_m$ and $\beta_j = 0$ for all j by the linear independence of $w_1, \dots, w_n$.

3. (10 pts) Let V be a vector space over the field F. Show that V and $\mathcal{L}(F, V)$ are isomorphic vector spaces.

   For a $v \in V$, define the map $T_v : F \to V$ by $T_v(\alpha) = \alpha v$. It is quite clear that T_v is linear. Now, define $\phi : V \to \mathcal{L}(F, V)$ by $\phi(v) = T_v$. To see that ϕ is linear, note that
$$T_{u+v}(\alpha) = \alpha(u + v) = \alpha u + \alpha v = T_u(\alpha) + T_v(\alpha),$$
and
$$T_{\lambda v}(\alpha) = \alpha(\lambda v) = (\alpha\lambda)v = (\lambda\alpha)v = \lambda(\alpha v) = \lambda T_v(\alpha)$$

by the usual properties of vector spaces and fields.

We will show ϕ has an inverse. Define $\sigma : \mathcal{L}(F,V) \to V$ by $\sigma(S) = S(1)$. If $S \in \mathcal{L}(F,V)$, let $v = \sigma(S) = S(1)$. Notice that for any $\alpha \in F$,

$$S(\alpha) = S(\alpha 1) = \alpha S(1) = \alpha v = T_v(\alpha).$$

Hence $S = T_v = \phi(v)$. This shows $\phi \circ \sigma = 1_{\mathcal{L}(F,V)}$. Now, if $v \in V$, then

$$\sigma(\phi(v)) = \sigma(T_v) = T_v(1) = 1v = v.$$

Hence $\sigma \circ \phi = 1_V$. Therefore ϕ is an isomorphism from V to $\mathcal{L}(F,V)$.

4. (10 pts) Prove the Linear Dependence Lemma: If V is a vector space and $v_1, \dots, v_m$ is a linearly dependent list of vectors in V, then there exists a $1 \le j \le m$ such that
   (a) $v_j \in \operatorname{span}(v_1, \dots, v_{j-1})$, and

   (b) if v_j is removed from $v_1, \dots, v_m$ then the span of the remaining list equals $\operatorname{span}(v_1, \dots, v_m)$.

   (To be clear, note that in (a), if $j = 1$, $v_1 \in \operatorname{span}(\emptyset)$ means $v_1 = 0$ since the span of the empty set/list is by definition $\{0\}$.)

   See Proposition 2.21 in your textbook.

5. (a) (2 pts) Define what a linear map is.

   If V and W are vector spaces over the same field F, then a map $T : V \to W$ is linear if it satisfies the following two properties:
   Additivity: $T(u+v) = T(u) + T(v)$ for all $u, v \in V$,

   Homegeneity: $T(\lambda v) = \lambda T(v)$ for all $\lambda \in F$ and for all $v \in V$.

   (b) (4 pts) Prove a linear map $T : V \to W$ is injective if and only if $\operatorname{null}(T) = \{0\}$.

   See Proposition 3.16 in your textbook.

   (c) (4 pts) Give an example of a linear map $T : V \to V$ that is injective but not surjective. Be sure to fully justify your example. (Hint: Is such a map possible if V is finite dimensional?)

   Let $V = F^\infty$ over F and let $T : V \to V$ be the right shift map, i.e.

   $$T(x_1, x_2, \dots) = (0, x_1, x_2, \dots).$$

   We verified in class that T is a linear map. Now, if $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$

   $$T(x) = T(y) \implies (0, x_1, x_2, \dots) = (0, y_1, y_2, \dots) \implies x_i = y_i \ \forall i \in \mathbb{Z}^+.$$

   Hence $x = y$. So T is injective. But T is clearly not surjective as any sequence whose first element is nonzero is not in the range.

   (d) (10 pts) Prove that Fundamental Theorem of Linear Maps: If V, W are vector spaces, V is finite dimensional, and $T \in \mathcal{L}(V,W)$ then

   $$\dim(V) = \dim(\operatorname{null}(T)) + \dim(\operatorname{range}(T)).$$

   See Proposition 3.22 in your textbook.

6. **Extra credit problem.** Let V be a vector space over some field F and U, W subspaces of V. In this problem, you will prove that $(U+W)/W \cong U/(U \cap W)$. I will help you by breaking this down into a few steps.

(a) (3 pts) First, define the map $T : U + W \to U/(U \cap W)$ by $T(u + w) = u + (U \cap W)$. Notice that while every vector $v \in U + W$ can be expressed as $u + w$ for some $u \in U$ and some $w \in W$, this expression is not unique in general. So depending on the choice you make for u and w, $T(v)$ could potentially give you different results. I.e. T may not be well-defined. Show that T is in fact well-defined.

Let $v \in U + W$ and suppose $v = u_1 + w_1 = u_2 + w_2$ are two different ways of writing v as an element of U plus an element of W. We need to show that either gives the same result when applying T to it. That is we want to show $u_1 + (U \cap W) = u_2 + (U \cap W)$. This is so if $u_1 - u_2 \in (U \cap W)$, that is $u_1 - u_2$ is both in U and in W. It is clear that $u_1 - u_2 \in U$. Notice

$$u_1 + w_1 = u_2 + w_2 \implies u_1 - u_2 = w_2 - w_1 \in W.$$

So $u_1 - u_2 \in U \cap W$ and hence $u_1 + (U \cap W) = u_2 + (U \cap W)$.

(b) (4 pts) Prove that T is a linear map.

Let $v_1, v_2 \in U + W$. Then $v_1 = u_1 + w_1$ and $v_2 = u_2 + w_2$ for some $u_1, u_2 \in U$ and $w_1, w_2 \in W$. Notice that $v_1 + v_2 = (u_1 + u_2) + (w_1 + w_2)$ where $u_1 + u_2 \in U$ and $w_1 + w_2 \in W$. So

$$T(v_1 + v_2) = u_1 + u_2 + (U \cap W) = u_1 + (U \cap W) + u_2 + (U \cap W) = T(v_1) + T(v_2).$$

Now, let $v \in U + W$ and $\lambda \in F$. Then $v = u + w$ and $\lambda v = \lambda u + \lambda w$ where $\lambda u \in U$ and $\lambda w \in W$. So

$$T(\lambda v) = \lambda u + (U \cap W) = \lambda(u + (U \cap W)) = \lambda T(v).$$

(c) (2 pts) Prove that T is surjective.

Let $u + (U \cap W) \in U/(U \cap W)$. Then $u = u + 0 \in U + W$ and $T(u) = u + (U \cap W)$. So any element of $U/(U \cap W)$ is in the range of T.

(d) (4 pts) Prove that $\text{null}(T) = W$.

First, if $w \in W$ then $w = 0 + w \in U + W$, so $T(w) = 0 + (U \cap W)$. This shows $W \subseteq \text{null}(T)$. Now, suppose $v \in \text{null}(T)$. Since $v \in U + W$, $v = u + w$. That $v \in \text{null}(T)$ means $T(v) = 0 + (U \cap W)$. But $T(v) = u + (U \cap W)$ as well. Hence

$$u + (U \cap W) = 0 + (U \cap W) \implies u - 0 \in (U \cap W) \implies u \in W.$$

Hence $v = u + w \in W$. This shows $\text{null}(T) \subseteq W$. So $\text{null}(T) = W$.

(e) (2 pts) Conclude that the map $\tilde{T} : (U + W)/W \to U/(U \cap W)$ defined by

$$\tilde{T}((u + w) + W) = u + (U \cap W)$$

is well-defined and is an isomorphism. Hence $(U + W)/W \cong U/(U \cap W)$.

To see that $\tilde{T}$ is well defined, we need to check that if $v_1, v_2 \in U + W$ such that $v_1 + W = v_2 + W$ then $\tilde{T}(v_1 + W) = \tilde{T}(v_2 + W)$. First, notice that for any $v \in U + W$, $\tilde{T}(v + W) = T(v)$. If $v_1 + W = v_2 + W$ then $v_1 - v_2 \in W = \text{null}(T)$, so

$$\tilde{T}(v_1 + W) - \tilde{T}(v_2 + W) = T(v_1) - T(v_2) = T(v_1 - v_2) = 0 + (U \cap W).$$

Hence $\tilde{T}(v_1 + W) = \tilde{T}(v_2 + W)$.
By Proposition 3.91 $\text{range}(\tilde{T}) = \text{range}(T) = W$ and hence $\tilde{T}$ is an isomorphism $(U + W)/W \to U/(U \cap W)$.