Math 413 Exam 1 Solutions

Oct 9, 2007

1. (10 pts) A prince picked a basketful of golden apples in the enchanted orchard. On his way home, he was stopped by a troll who guarded the orchard. The troll demanded payment of one-half of the apples plus two more. The prince gave him the apples and set off again. A little further on, he was stopped by a second troll, who demanded one-half of the apples the prince now had plus two more. The prince paid him and set off again. Soon, a third troll stopped him and demanded one-half of his remaining apples plus two more. The prince paid him and sadly noted that had only five golden apples left. How many apples had he picked?

   Give a solution of the above problem that is not based on guess-and-check. Make sure you carefully justify each step of your method.

   Suppose the prince started out with x apples. After meeting the first troll, he would have half his apples less two left. That is $x/2-2$. After meeting the second troll, he would have half of this less two left, which is $(x/2-2)/2-2$ left. Similarly, after the meeting the third troll, he'll have $((x/2-2)/2-2)/2-2$ apples left. So

$$\frac{1}{2}\left(\frac{1}{2}\left(\frac{x}{2}-2\right)-2\right)-2=5$$
$$\frac{x}{8}-\frac{1}{2}-1-2=5$$
$$x-28=40 \implies x=68$$

   So the prince must have started out with 68 apples.

   Indeed, we can easily check this answer. If he does start out with 68 apples, he gives $34+2=36$ to the first troll and keeps 32. Then he gives $16+2=18$ to the second troll and keeps 14. Finally, he gives $7+2=9$ to the third troll and ends up with 5.

2. (15 pts) Eight adults and two children need to cross a river. A small boat is available that can hold either one adult, or one child, or two children. Everyone can row the boat, but the boat cannot cross the river by itself without anyone in it. How many one-way trips does it take for all of them to cross the river?

   (a) Give a correct solution to the above problem. Fully justify every claim you make.

   We could start by having an adult row the boat to the other side. But the boat now has to come back for the remaining people. The only way it can come back is if the same adult that went over with it brings it back. This is not helpful as it adds 2 one-way trips to the count without accomplishing anything. For the same reason, we don't want to start by letting a child row the boat over to the other side. So the only reasonable way to start is to send two children over and have one of them bring the boat back.

   Once the boat is back, we can choose between sending an adult over to the other side or sending the child that brought the boat back again. The latter only gets us back to the same situation we were in 2 trips ago. So we'll want to send an adult across. Once on the other side, the adult gets out and the child on that side of the river can bring the boat back. Note that this is the only thing we can do to make progress in moving people to the other side.

   We can now keep repeating the same procedure. It takes 4 trips to transport and adult to the other side and have the boat back on this side of the river this way. It will take 32 trips for all the adults to get to the other side. Now we still have the two children

on this side. They can both cross to the other side in trip. So the total number of trips required is 33.
We can't do better than this as any other step, such as having an adult or both children bring the boat back, would only cost additional trips without doing us any good.

(b) What if there were x adults and y children? Find the number of one-way trips required to cross the river in terms of x and y. (Hint: could x and y be such that the problem has no solution? If so, don't forget to discuss such cases too.)

First note that if $y \geq 2$ then the analysis in part (a) still applies and the best we can do is transport each adult across with 4 trips. So we will need $4x$ trips just for the adults. Now we can take two children across in one trip, but if $y > 2$ the boat has to return for the remaining children. This means one of the children already on the other side must bring it back. Once this child is back with the boat, another child can hop in and they can row to the other side. Now we just keep repeating this procedure until all the children are on the other side. So the first two children go across in one trip, then each remaining child takes two additional trips to bring over. This means we need $2(y-2)+1 = 2y-3$ trips to transport the children across. So the total number of trips needed is $4x + 2y - 3$.
We cannot do better than this. We could rearrange the order of the trips, as in take the children across first and then the adults, or alternate, but this wouldn't change the total number of trips required.
If $y < 2$ then we have the following cases.

$y = 1$, $x > 0$: In this case there is no solution. Whether we let the child cross first, or one of the adults, the same person has to bring the boat back. We are back to where we started without any progress.

$y = 1$, $x = 0$: In this case the child crosses in one trip and we are done.

$y = 0$, $x > 1$: Again, there is no solution. We can only take one adult to the other side.

$y = 0$, $x = 1$: In this case, the adult crosses in one trip and we are done.

$y = 0$, $x = 0$: The task is accomplished in 0 trips.

Some of you made the following argument. Let's make a table about the number of children and how many trips are needed to move them to the other side:

Number of children	Trips needed
2	1
3	3
4	5
5	7

We can see that the formula $2y - 3$ fits the data in this table. Therefore it takes $2y - 3$ trips to move y children to the other side of the river.
This is an incomplete argument. We really have no way of knowing that formula will continue to work for other numbers of children. In fact, you can try that the formula $y^4 - 10y^3 + 35y^2 - 48y + 21$ also fits the data in the table. One could even guess that the number of trips required is $2y + \sin(\pi y) - 3$. This would even continue to work for other values of y as long as y is always an integer. But it would be difficult to explain the role of the sine function. While the table gives you a chance to guess the right formula, you still need to justify that formula.

Remember what we said in class about the essential role deductive reasoning plays in mathematics. You should always deduce formulas from the principles of the problem.

3. (15 pts)
   (a) List three aspects of algebraic thinking, as we discussed them in class. Find an example of each aspect in your solutions to one of the first two problems on this exam. If an aspect is not apparent in any of your solutions above, explain how it could be introduced.

   Generalization, abstraction: In problem 2.(b) we generalize what we did with 8 adults and 2 children to x adults and y children.

   Patterns, functions: In the solution to problem 1, we construct a function which gives the number of apples left if the original number of apples was x.

   Reversibility: In problem 1, we solve an equation by reversing the operations done to x on one side of that equation.

   (b) Argue why guess-and-check does not qualify as a complete solution to the golden apple problem (problem 1 on this exam).

   The biggest problem with guess-and-check is that it does not guarantee finding the solution. One may make many wrong guesses and never try the right answer. Another issue is that if the problem has no solution, guess-and-check won't tell you. You'll just keep guessing wrong answers until you give up. Finally, if the problem has several correct answers, guess-and-check won't tell you how many, and you are likely to stop guessing after you find one, content that you've found "the answer." Finally, guess-and-check doesn't lend itself to generalization. While it may let you solve this particular problem, it doesn't teach you how to solve related problems.

   (c) What question could you ask a student who tries to solve the golden apple problem by guess-and-check to steer them toward a more algebraic solution?

   There are many good questions. One is "What would you do if the prince had x apples left in the end?"

4. (10 pts)
   There are 200 lockers in one hallway of the King School, one for each student. At the beginning of the day, all of the lockers are closed.

   To express their joy at being at school, the students play the following game. First, all 200 of them line up. Then the first student in the line runs down the row of lockers and opens every locker. Next, the second student runs the down the row and closes every other door (starting with #2). The third student runs down the row and changes the state of every third locker (starting with #3): he (or she) opens those doors that are closed and closes those that are open. This goes on until the 200th student passes.

   (a) Explain the connection between the number of divisors of a number and the final state of the corresponding locker.

   It is easy to see that the n-th student will change the state of a locker if and only if n divides the number of the locker. Hence the number of times the state of a locker changes is the number of divisors of the number on it. Lockers which correspond to numbers with an even number of divisors end up closed, whereas lockers which correspond to numbers with an odd number of divisors end up open.

(b) Which lockers are open at the end of this process? Carefully explain why. Your explanation should also address why the other lockers end up closed.

As we said in the answer to part (a), the lockers that end up open are exactly the ones whose number has an odd number of divisors. Such numbers are the perfect squares. This is because for a number n, one can pair up each divisor k of n with n/k. So the divisors of a number come quite naturally in pairs. Most such pairs will consist of two different numbers. The one exception to this is when $k = n/k$, that is when $n = k^2$. Hence the number of divisors of a number n is even unless n is a perfect square. This shows that the lockers that end up open are exactly those whose number is a perfect square, namely 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196.

5. (10 pts) **Extra credit problem.**

Five pirates have raided a ship and stolen 100 gold coins. They now have to divide up the loot. The pirates are all extremely intelligent, mean, and selfish.

They agree that the captain will propose a distribution of the loot. All pirates (including the captain) vote on the proposal, and if half the crew or more go "Aye", the loot is divided as proposed. No pirate would be willing to take on the captain without superior force on their side.

If more than half the crew–which includes the captain–vote against the captain's proposal, there is a mutiny, and all pirates turn against the captain and kill him. Then the pirates elect the most senior pirate captain and start over again with distributing the loot.

What is the maximum number of coins the captain can keep without risking his life?

I don't want to give away the solution because I am hoping that you are intrigued enough by this problem that you will continue to think about it. But here are a few ideas you may want to think about. Be warned that some of these ideas may lead you in the wrong direction, but if you think about them, eventually you should learn enough about the problem that you can find your own solution.

- The captain only needs two allies to have his proposal accepted. So all he has to do is suggest an arrangement that pleases two of the lower-ranking pirates.
- To decide whether to vote for or against the captain's proposal, a pirate will consider what is likely to happen if the captain is killed. If the captain offers him more coins than he is likely to get with the captain dead, then the pirate will be pleased with the captain's proposal. Also, if the pirate thinks that he himself is likely to be killed if he becomes captain eventually, then he has a compelling reason to vote for the captain's proposal, no matter how little money it awards him.
- The captain is selfish, so he will want to keep as many coins to himself as he possibly can without putting himself in danger. The fairness of splitting the gold evenly among the 5 pirates holds no particular appeal to him. On the other hand, the lower-ranking pirates may not be impressed with the fairness of an even split either. They are not going to vote for the captain's proposal just because it is fair and just, if they think they can get more coins by killing the captain.
- If the four lower-ranking pirates kill the captain, then they have 100 coins to distribute among the four of them. Does this suggest that they are better off killing the captain no matter what he proposes?
- Continuing the above reasoning, wouldn't it always make sense for the lower-ranking pirates to vote against the current captain so that fewer of them remain to share the loot? How far can you take this reasoning?