Math 413 Exam 2 Solutions

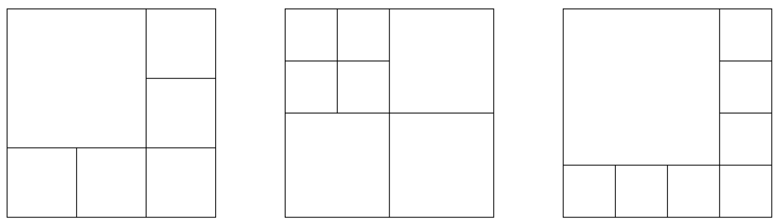
Nov 8, 2007

1. (10 pts) Recall our work in class with Cuisenaire rods. How many Cuisenaire trains of length n can you build out of k Cuisenaire rods, if you distinguish between the front and the back of the train? Why? Carefully justify your answer.

Think about the problem backwards (reversibility!). Instead of combining shorter rods to make a rod of length n, start with a rod of length n and cut it into k pieces. You need to make $k-1$ cuts and there are $n-1$ places to cut. So you are choosing $k-1$ out of the $n-1$ possible cuts. There are $\binom{n-1}{k-1}$ ways to do this. We get every possible train this way and each choice results in a different train.

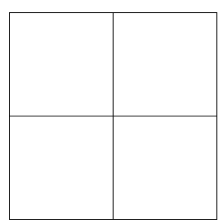
2. (10 pts) In this problem, you will consider dividing a square into a number of smaller squares (of possibly different sizes).

(a) Show that you can divide a square into 6, 7, or 8 smaller squares.



(b) Use your result in part (a) to argue that a square can always be divided into n smaller squares if $n \geq 6$.

Notice that you can always cut a square into four smaller squares like this:



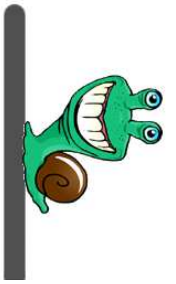
This replaces 1 square with 4, and so increases the number of squares by 3. So we have three cases depending on n:

$n \geq 9$ **is divisible by** 3**:** Start cutting the original square into 6 squares as in part (a), then keep cutting some of the squares into 4 squares. This will give you $9, 12, 15, \ldots$ squares. You will reach n eventually.

$n = 3k+1$ **for some integer** $k \geq 3$**:** Start cutting the original square into 7 squares as in part (a), then keep cutting some of the squares into 4 squares until you have n squares.

$n = 3k+2$ **for some integer** $k \geq 3$**:** Start cutting the original square into 8 squares as in part (a), then keep cutting some of the squares into 4 squares until you have n squares.

These three cases along with the figures in part (a) cover all numbers $n \geq 6$.



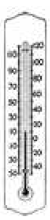
3. (10 pts) A snail started crawling up a pole from the bottom. It went up 4 feet each day and slid back 3 feet each night.
   (a) How long would the snail take to go to the top if the pole is 5 feet high?

   The first day the snail will get to a height of 4 feet. Then it will slide down to 1 foot overnight. Starting from there, the snail will reach the top of the pole by the end of the second day. So it will take 2 days.

   (b) How long would the snail take to go to the top if the pole is m feet high?

   The snail will reach the top of the pole on the day when it starts at a height of $m - 4$ feet. The snail advances $4 - 3 = 1$ foot in one whole day and night. Hence it will start the day at a height of $m - 4$ feet after $m - 4$ days and nights. Then it takes one more day to reach the top. Therefore the answer is $m - 3$ days.

4. (10 pts) The following two problems are independent of each other. The answers may be different.



   (a) On the 12th floor of the Acme Building, Guy Noir, private eye was looking at a photograph taken last spring. The picture showed a desk in front of a window. On the desk was thermometer-clock device, one that showed the outside temperature. "Curious," Mr. Noir noted, "the date on the clock is 04.05.07 and the time is 1:03:03 PM. If I divide the temperature by 4, I get a remainder of 1. If I divide it by 5, I get a remainder of 3. If I divide it by 7, the remainder is 3." What temperature did the thermometer show? How can you tell?

   Let x be a number which satisfies the conditions in the problem. Then $x - 3$ is divisible by both 5 and 7, hence $35|x - 3$. So x gives a remainder of 3 when divided by 35. Such positive numbers are $3, 38, 73, \ldots$. Among these, 73 is the first which gives a remainder of 1 when divided by 4. Of course, 73 is not the only such number. If y also satisfies these conditions, then $y - 73$ must be divisible by 4, 5, and 7 and therefore by $\mathrm{lcm}(4, 5, 7) = 140$. Indeed, adding multiples of 140 to 73 does not change the remainders of division by 4,5, or 7. But $73 + 140 = 213$ is already way too hot and $73 - 140 = -67$ is too cold to be realistic in most places on earth in April. So the temperature was almost certainly 73°F. There is a remote possibility that the picture showed a desk at a polar research station, in which case the temperature could have been $-67°$ F.
   BTW, the problem does not say what units the temperature was measured in. If they were °C, then 73°C is unrealistically hot. So the temperature would have been -67°C, which is possible but unlikely. It would again place the picture at a polar research station. I suppose Kelvins could have been used too, but I don't think one could find a thermometer-clock which measures temperature in K.

daffodil

   (b) On the 12th floor of the Acme Building, Guy Noir, private eye was looking at a photograph taken last spring. The picture showed a desk in front of a window. On the desk was thermometer-clock device, one that showed the outside temperature. Visible through the window in the distance was a park with a few yellow daffodils amid patches of snow, harbingers of spring after a cold Minnesota winter. "Curious," Mr. Noir noted, "the date on the clock is 04.05.06 and the time is 1:03:03 PM. If I divide the temperature by 4, I get a remainder of 1. If I divide it by 5, I get a remainder of 3. If I divide it by 6, the remainder is 3." What temperature did the thermometer show? How can you tell?

   Using the same logic as in part (a), the temperature should be of the form $30k + 3$. We can consider $3, 33, 63, \ldots$. In fact, 33 satisfies the conditions of the problem. Once again,

other numbers that would fit the conditions would be 33 plus multiples of $\text{lcm}(4, 5, 6) = 60$. But 93°F and above are too hot for patches of snow and −27°F and below are too cold for daffodils. So the temperature must have been 33°F.
Again, the temperature could have been measured in °C. But neither −27°C nor 33°C are consistent with the conditions in the park.

If you are curious how to do these problems using Diophantine equations, here it is for part (a). Given the conditions, we are looking for a number t which is of the form $4x + 1$, $5y + 3$, and $7z + 3$ for some integers x, y, z at the same time. Setting these equal gives us two independent equations:

$$4x + 1 = 5y + 3$$
$$5y + 3 = 7z + 3$$

Using the second equation, we get

$$5y = 7z \implies y = 7k \text{ for some integer } k.$$

This is because $7 \nmid 5$, so $7|y$. Substitute this into the first equation to get

$$4x + 1 = 35k + 3 \implies 4x = 35k + 2 \implies x = \frac{35k + 2}{4} = 8k + \frac{3k + 2}{4}.$$

Since x must be an integer, we are looking for k such that $4|3k+2$. So $3k+2 = 4m$ for some integer m. Hence

$$k = \frac{4m - 2}{3} = m + \frac{m - 2}{3}.$$

That is $3|m - 2$, so $m = 3n + 2$ for some integer n. We can now get back to t:

$$k = \frac{4m - 2}{3} = \frac{4(3n + 2) - 2}{3} = \frac{12n + 8 - 2}{3} = 4n + 2$$
$$x = \frac{35k + 2}{4} = \frac{35(4n + 2) + 2}{4} = \frac{140n + 70 + 2}{4} = 35n + 18$$
$$t = 4x + 1 = 4(35n + 18) + 1 = 140n + 73.$$

You just need to consider for what integers n this gives a realistic temperature.

5. (10 pts) What are three main traits of algebraic thinking? Discuss how each trait appears in either the question of problem 2 on this exam or your solution to that question. If a trait does not appear, say so.

The three traits are
- generalization and abstraction,
- patterns and functions,
- reversibility.

Part (b) of the problem *generalizes* part (a). Dividing a square into 4 smaller squares increases the number of squares by 3 and establishes a *pattern* you can use to solve the problem for larger numbers. Notice that in part (b), we give a *function* whose input is an integer $n \geq 6$ and whose output is a division of the square into n smaller squares. Reversibility is less obvious. But one could argue that we solve the problem of dividing a square into n smaller squares by reverting to dividing it into 6, 7, or 8 squares and then further subdividing squares into 4 smaller squares as needed.

6. (10 pts) **The pirates return in this extra credit problem.**

The same five pirates you know from the last exam have again raided a ship. To their great disappointment, they have only looted one gold coin this time. They now have to decide what to do with the gold. As usual, they agree that the captain (who is one of the five pirates) will propose what to do. All pirates–including the captain–vote on the proposal, and if half the crew or more go "Aye", the loot is divided as proposed, as no pirate would be willing to take on the captain without superior force on their side.
If more than half the crew–which includes the captain–vote against the captain's proposal, there is a mutiny, and all pirates turn against the captain and kill him. Then the pirates elect the most senior pirate captain and start over again with distributing the loot.
The pirates are all extremely intelligent, mean, and selfish. They are also upset over the meager size of the loot. Each pirate's priorities are in order:
   1. his life,
   2. money,
   3. killing fellow pirates (so next time, the loot has to be divided among fewer people).

That is if a pirate has a choice between two outcomes in which he gets the same amount of money, he will choose the one in which more of his fellow pirates are killed.
What distribution can the captain propose if he wants to survive? Is there such a distribution? Think carefully!

I don't want to tell you how to solve the extra credit problem because I hope that the problem will continue to intrigue you. You can start off by considering what would happen if only two pirates remained. Which of the two would get the coin? What does this suggest will happen if at one point there are three pirates left? And so on.