# Math 413 Exam 1 Solutions

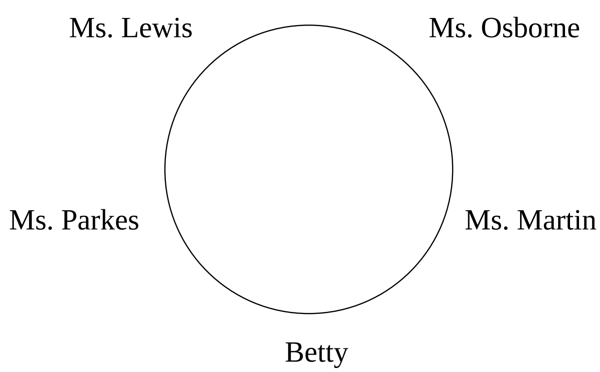
Feb 22, 2010

1. (10 pts) Five women have lunch together seated around a circular table. Ms. Osborne is sitting between Ms. Lewis and Ms. Martin. Ellen is sitting between Cathy and Ms. Norris. Ms. Lewis is between Ellen and Alice. Cathy and Doris are sisters. Betty is seated with Ms. Parkes on her left and Ms. Martin on her right. Match the first names with the surnames.

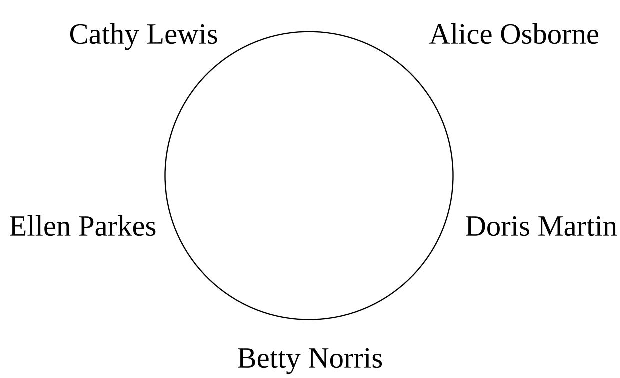
Let's number the statements for easier reference:

(a) Ms. Osborne is sitting between Ms. Lewis and Ms. Martin.
(b) Ellen is sitting between Cathy and Ms. Norris.
(c) Ms. Lewis is between Ellen and Alice.
(d) Cathy and Doris are sisters.
(e) Betty is seated with Ms. Parkes on her left and Ms. Martin on her right.

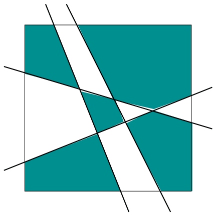
By 5, Ms. Parkes is on Betty's left and Ms. Martin is on Betty's right. By 1, Ms. Osborne is sitting next to Ms. Martin, but not next to Ms. Parkes. So Ms. Osborne is not Betty, but is in fact sitting on Ms. Martin's right, followed by Ms. Lewis. Here is the current seating arrangement:



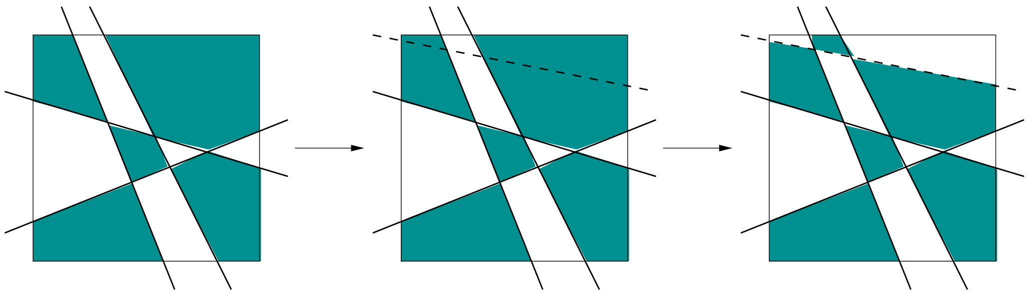

Since all other last names are accounted for, Betty must be Ms. Norris. By 2 and 3, Ellen is next to both Betty Norris and Ms. Lewis. This makes Ellen the same as Ms. Parkes. Hence Cathy is Ms. Lewis and Alice is Ms. Osborne. The only remaining first name is Doris, and this must be Ms. Martin's first name. So here is the final arrangement:



2. (10 pts) Take a square and draw a straight line across it. Draw several more lines in any arrangement so that all the lines cross the square, and the square is divided into several regions. The task is to color the regions in such a way that adjacent regions are never colored the same. (Regions having only one point in common are not considered adjacent.) How few different colors are needed to color such an arrangement?

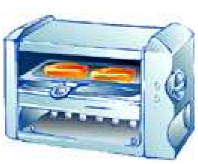


For simplicity, let's say the two colors are blue and white. Let's draw the lines one by one and always color the square with two colors as we go. Notice that the statement is true when you draw no lines at all (actually one color is sufficient in this case). Now do the following. Every time you draw a new line, it will cut through some regions, dividing them into two parts. These two adjacent parts will now have the same color. We will fix this problem by choosing one side of the line and reversing the colors of all the regions on that side of the line. Notice that this fixes the problem we have had: two newly adjacent regions on the two sides of the line will now have different colors. Here is an illustration of this with the last line drawn as a broken line:



Reversing the colors does not introduce any new color conflicts either, as any regions that were adjacent prior to drawing the line, and hence had different colors, will still have different colors, only those colors will now be switched. This shows that no matter how many lines we draw, we can always color the resulting regions with at most two colors.

3. (10 pts) Three slices of bread are to be toasted under a grill. The grill can hold two slices at once but only one side is toasted at a time. It takes 30 seconds to toast one side of a piece of bread, 5 seconds to put a piece in or take a piece out and 3 seconds to turn a piece over. What is the shortest time in which the three slices can be toasted?



We will assume that it is possible to multitask, as in taking one piece of bread out while another is toasting, or flipping two slices at the same time. Also, for ease of communication, we will refer to the two spaces in the grill as the left and right sides of the grill.

Here is a scheme for toasting the bread in 113 s. Let's number the three slices for easier reference. Insert slices #1 and #2 on the left and right sides of the grill respectively. This takes 5 s. 30 s later, flip slice #1 while removing slice #2. Insert slice #3 on the right. So slice #3 starts toasting at the 45th sec. At the 68th sec, remove slice #1, which is now fully done. Insert slice #2 on the left so its untoasted side is now being toasted. While doing this, flip slice #3 at the 75th sec. So the second sides of slices #2 and #3 start toasting at the 78th sec. 30 s later they are both done. Remove them simultaneously. Now all three slices are fully done and the total time elapsed is 113 s.

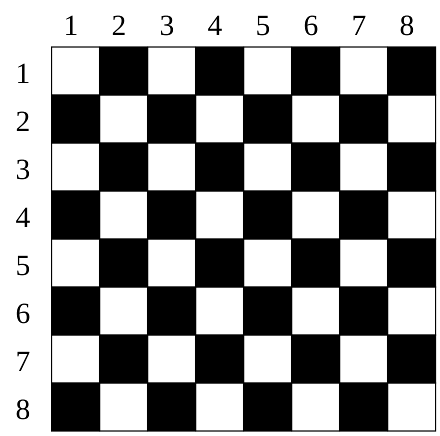
We need to prove this is the best we can do. Toasting the six sides takes 180 s altogether. Putting the three slices in and taking them out takes another 30 s. Finally, each slice needs to be flipped at least once, which is another 9 s. So the entire job needs 219 s. Since the grill can handle two slices simultaneously and we have assumed that multitasking is possible, this time can be cut in half under ideal conditions, so the minimum time required is actually 109.5 s.

Our computation of the minimum assumed that once a slice is inserted into the grill, flipping it once is the only thing that happens to it before it is fully done and removed. The problem with this is that it requires toasting two slices of bread on the same side of the grill, one after the other, which takes a total of 144 s (30 s for toasting each of four sides, 10 s for inserting and removing each of the two slices, and 3 s for flipping each of the two slices). We can obviously do better than this, but only if we replace at least one flipping by a remove-insert operation. That replaces a 3 s procedure with a 10 s procedure, increasing the total time required to 226 s. Using both sides of the grill and multitasking, that still takes at least 113 s. In fact, we know we can complete the task in 113 s. Now we have shown that it cannot possibly be done in less time.

4. (10 pts each)

   (a) If you count carefully, you can find 204 squares on an ordinary $8 \times 8$ chessboard. Justify this result.

   Let's number the squares of the chessboard as below.




   We can have $1\times 1, 2\times 2, \dots, 8\times 8$ squares on the board. Let us consider where the upper left corners of these squares can be on the board. Obviously, the upper left corner of an $8 \times 8$ square can only be in the $(1, 1)$ position. As for $7 \times 7$ squares, their upper left corners could be in the $(1, 1), (1, 2), (2, 1), (2, 2)$ positions. Anywhere else, and the $7 \times 7$ square would not fit on the board. This gives $4 = 2^2$ different $7 \times 7$ squares. Continuing this way, we get $9 = 3^2$ positions for the upper left corners of $6 \times 6$ squares:

$$(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3).$$

   Continuing this way, we get the following chart:

| Size of square | $1 \times 1$ | $2 \times 2$ | $3 \times 3$ | $4 \times 4$ | $5 \times 5$ | $6 \times 6$ | $7 \times 7$ | $8 \times 8$ |
|---|---|---|---|---|---|---|---|---|
| Number of squares | $8^2$ | $7^2$ | $6^2$ | $5^2$ | $4^2$ | $3^2$ | $2^2$ | $1^2$ |

   Hence the total number of squares is $1^2 + 2^2 + \cdots + 8^2 = 204$.

   (b) How many squares are there on an $n \times n$ chessboard?

   The argument in part (a) readily generalizes to the $n \times n$ chessboard. It gives $n^2$ $1 \times 1$ squares, $(n-1)^2$ $2 \times 2$ squares, etc. In general, the number of $k \times k$ squares is $(n-k)^2$.

So the total number is

$$n^2 + (n-1)^2 + \cdots 1^2 = \sum_{k=1}^{n} k^2.$$

This happens to be equal to $\frac{n(n+1)(2n+1)}{6}$, which can be proved by induction, if you know what that is.

5. (a) (4 pts) What are specializing and generalizing?

   Specializing means looking at specific examples of a general statement, as in replacing a variable(s) with a concrete value(s) replacing a broad condition with a narrower one. Generalizing means finding a pattern by looking at specific examples and applying that pattern to guess, or explain, or prove what happens in a more general setting. It also means relaxing the conditions of a problem to lead to a more general statement.

   See the top of p. 24 in Thinking Mathematically for other ideas.

   (b) (6 pts) Describe in some detail the role specializing and generalizing play in solving math problems.

   Specializing helps with understanding how and why a general statement holds by allowing to look at it in concrete cases. It is dominant in the initial (entry) phase of problem solving when we need to build familiarity with what the problem is asking for. But it is also useful later on to test the validity of our ideas and conjectures. It can serve to gather evidence for a claim or counter-evidence against one.
   Generalizing means expanding a problem and/or its solution from specific cases to a broader setting. Generalization is involved in observing patterns, forming conjectures, and expanding an argument. Making general statements about how mathematical objects behave and in what setting mathematical principles apply is one of the major goals of math. Generalization dominates in the later stages of problem solving (attack and review phases).

6. (10 pts) **Extra credit problem.** Mr. and Mrs. Jones are entertaining four other couples. During the evening, many handshakes occur but no one shakes hands with their spouse (or with themselves of course). At the leave taking, Mr. Jones asks everyone (each guest and Mrs. Jones) how many hands they had shaken, and everyone tells him a different number. How many hands did Mr. and Mrs. Jones shake?

   Here is a sketch of the argument:
   - One person can shake hands with at most 8 people.
   - Mr. Jones asks 9 people and they all give him different numbers. Therefore each of the numbers $0, 1, \ldots, 8$ are used exactly once.
   - Whoever shook hands with 8 people, shook hands with everyone but their spouse. So none of the people not married to this person can claim to have shaken hands with nobody. Therefore the only person who can say they shook 0 hands must be married to the one who said 8.
   - Eliminate the 8-0 couple from the picture and consider the number of handshakes among the remaining group of four couples. Run through a similar analysis to eliminate a 7-1 couple.
   - Continue doing this until the only couple remaining is the 4-4 couple. This must be Mr. and Mrs. Jones, otherwise when Mr. Jones asked everyone about the handshakes,

he would have been told 4 by two different people. Conclude that Mr. and Mrs. Jones each shook hands with 4 other people.

This problem was a Car Talk Puzzler on Jan 24, 2005. In fact if you need more details, you can find a fully illustrated solution at

`www.cartalk.com/content/puzzler/transcripts/200504/answer.html`.