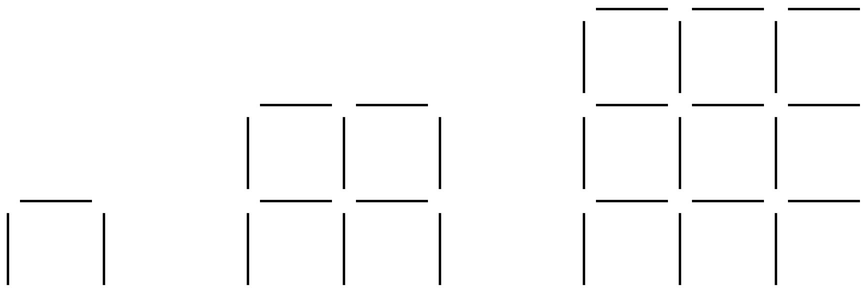
Math 413 Exam 2 Solutions

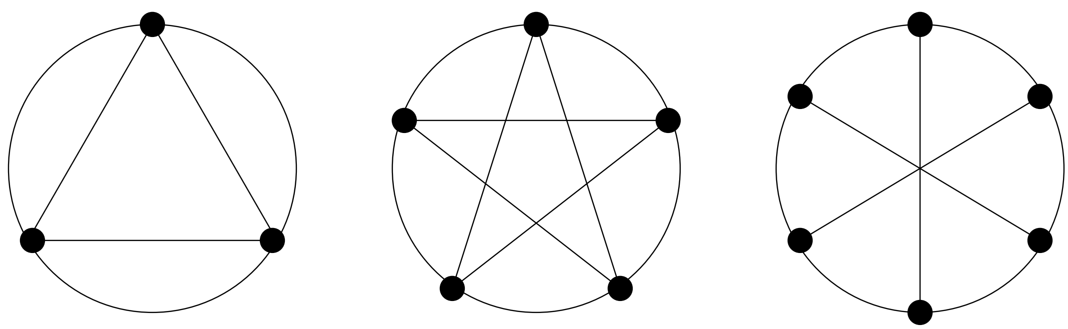
Mar 22, 2010

1. (10 pts) How many matches are required to make N^2 unit squares in a square array as in the following sequence?



One way to count them is to notice that there are $N+1$ vertical lines and $N+1$ horizontal lines of matches. Each vertical and each horizontal line consists of N matches. Therefore $(N+1)N$ matches are needed for the vertical lines and the same number for the horizontal lines. That is $2N(N+1)$ matches.

2. (10 pts) A number of pins are placed around a circle. A thread is tied to one pin, and then looped tightly around a second pin. The thread is then looped tightly round a third pin so that the clockwise gap between the first and second pin is the same as the clockwise gap between the second and third pin, as illustrated in the example:



3 pins, gap of 1   5 pins, gap of 2   6 pins, gap of 3

The process is continued, always preserving the same clockwise gap until the first pin is reached. If some pin has not yet been used, the process starts again.

Five pins with a gap of two use just one thread, while six pins with a gap of three use three threads. How many pieces of thread will be needed in general?

Let p be the number of pins and g the gap. Looking at a few examples will suggest that the correct number of threads is the greatest common divisor of p and g. We will now prove this conjecture.

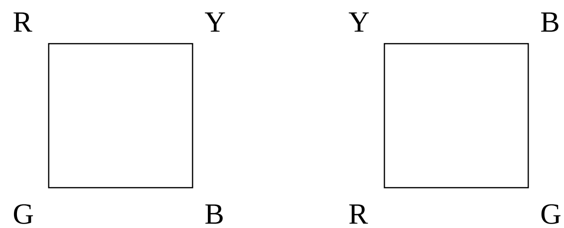
For convenience, let's call the pin where we start with the thread the 0th pin. For the thread to return to the starting pin, it has to pass p or $2p$ or $3p$, etc pins. That is the thread returns to the starting pin each time it has passed a multiple of p pins. But we skip some of the pins in the process and in fact the thread is wound around the g-th, $2g$-th, $3g$-th, etc pins only. That is it reaches only pins numbered a multiple of g. The first time we return to the starting pin is when this multiple of g is also a multiple of p. That is after $\mathrm{lcm}(p, g)$ pins (some of which are skipped pins). In the process, we wind the thread around $\text{lcm}(p, g)/g$ pins. So each piece of thread is wound around $\text{lcm}(p, g)/g$ pins. Since we have p pins, we need

$$\frac{p}{\text{lcm}(p, g)/g} = \frac{pg}{\text{lcm}(p, g)} = \gcd(p, g)$$

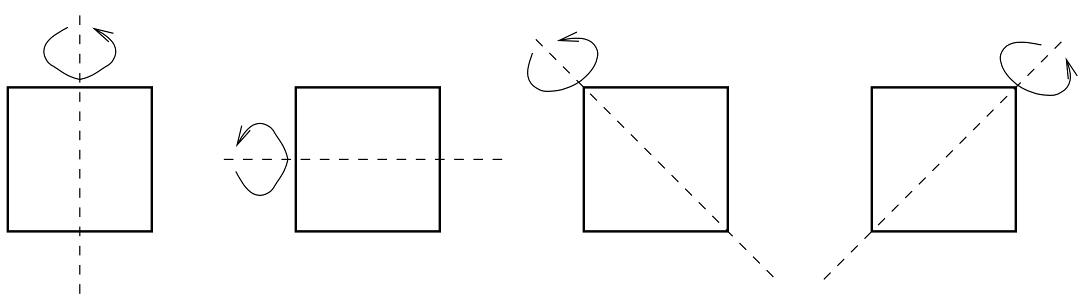
pieces of thread.

3. (5 pts) How many ways are there to color the four vertices of a square red, green, blue, and yellow if each color is to be used exactly once?

   We can color the first vertex one of four colors, the second vertex one of three reamining colors, the third vertex one of two remaining colors, and we have no choice left for the fourth vertex. That is $4! = 24$ choices. But notice that a square has no first vertex, second vertex, etc. In fact, its vertices are indistinguishable because of its symmetries. So for example the following two colorings are the same because we could rotate the first square to look exactly like the second:



   If we live in a 2-dimensional universe, then the square has 4 rotational symmetries (rotations through 0°, 90°, 180°, and 270° about the center). These rotations make 6 groups 4 colorings equivalent to one another. So there really are only 6 nonequivalent colorings. If we also allow rotations of the square outside the plain in which is lies, that gives four more rotational symmetries:



   I.e. the 24 colorings are really 3 groups of 8 equivalent colorings each. So in this case there are only three different ways to color the square.

4. (a) (3 pts) What is a conjecture and what role do conjectures play in solving mathematical problems?

   A conjecture is a statement that is believed to be true but has not been proved. The role of conjectures in problem solving is that it is through stating conjectures that we clarify what it is we want to prove.

   (b) (2 pts) How are conjectures made?

   Conjectures are made by looking at special cases of a problem and gathering intuition and initial evidence to have a good enough idea what statement is likely to be true.

(c) (10 pts) In class, we conjectured that every positive integer that is a not a power of 2 can be expressed as a sum of consecutive positive integers. Prove this conjecture.

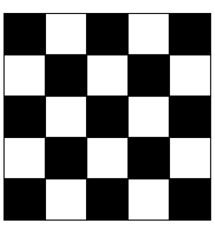
First, notice if n is a positive integer that is not a power of 2, then $n = 2^k m$, where $k \in \mathbb{Z}^{\geq 0}$ and m is some odd positive integer different from 1.
We will start by expressing m as a sum of consecutive positive integers. Since m is odd (and not 1), m can be written as $m = 2j + 1$ where j is some positive integer. Hence $m = j + (j + 1)$, which is a sum of two consecutive positive integers.
Now, if $k = 0$, then $n = m = j + (j + 1)$ and we are done. If not, then write

$$\begin{aligned}
&(j - 2^k + 1) + (j - 2^k + 2) + \cdots + j + \\
&\qquad (j + 1) + (j + 2) + \cdots + (j + 2^k) = \frac{(j - 2^k + 1 + j + 2^k)(2 \cdot 2^k)}{2} \\
&\qquad\qquad = (2j + 1)2^k = 2^k m = n
\end{aligned}$$

using the usual method of evaluating such a sum. So n is certainly a sum of consecutive integers. Some of the integers at the beginning of this sum may be negative, but if so, these will cancel with other positive integers later on. This will still leave some positive integers at the end, because j is positive, so anything after j is also positive. The result is a sum of consecutive positive integers equal to n.
Here is an example.

$$\begin{aligned}
56 &= 2^3 \cdot 7 \\
7 &= 3 + 4 \\
56 &= (-4) + (-3) + (-2) + \cdots + 10 + 11 \\
&= \underbrace{(-4) + (-3) + (-2) + (-1) + 0 + 1 + 2 + 3 + 4}_{=0} + 5 + \cdots + 10 + 11 \\
&= 5 + 6 + \cdots + 10 + 11
\end{aligned}$$

5. (10 pts) **Extra credit problem.** A knight is placed on a square of a 5×5 chessboard and moves according to the usual rule in chess. Is it possible to move the knight on the chessboard so that it visits each square exactly once and eventually returns to the square it started on, visiting this square, and only this square, for a second time?



This is not possible. Notice that the way the knight moves, it jumps from white square to black square and vice versa. That is it alternates between black and white squares. There are 13 black squares and 12 white squares on the board. If we want to visit them all, we need to start on a black square and end on a black square. (If we started on white, then after 23 moves, we'd be on a black square and there would be no white square to jump to from there.) But then there is no way to return from the last square to the initial square because that it also black.