Math 413 Final Exam Solutions

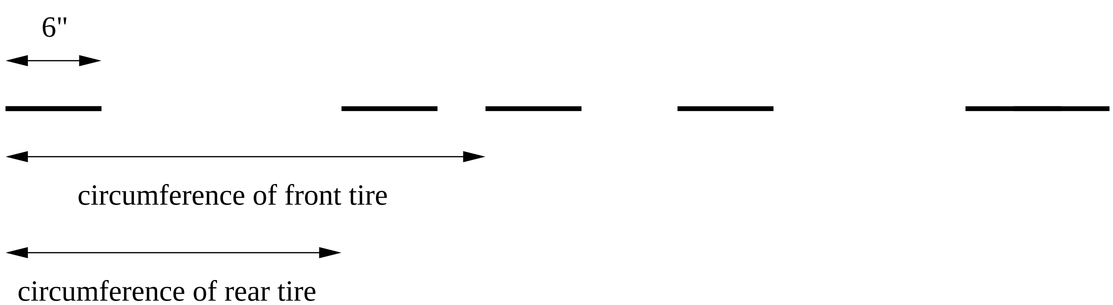
May 19, 2010

1. (10 pts) Once while riding on my bicycle along a path I crossed a strip of wet paint about 6 inches wide. After riding a short time in a straight line I looked back at the marks on the pavement left by the wet paint picked up on my tires. What did I see?

Since I rode in a straight line, the front and rear tires both followed the same straight path. It is reasonable to assume that the circumference of each tire is more than 6 inches, otherwise the bicycle would have to be very small. Let's look at the front tire first. It was stained by the paint on a 6 inch long section, and every time this section would come around to the ground, the tire would leave a 6 inch strip of paint on the ground. So it will make a broken line of 6 inch painted pieces one tire circumference apart, like this:

Let's now look at the rear tire. One modern bicycles, both tires tend to have the same diameter. The rear tire crosses the paint in exactly the same location as the front. So it will leave paint marks in exactly the same places as the front tire. That is the marks on the pavement still look exactly the same as the picture above.

If I have an unusual bicycle with different size tires in the front and the back, then the picture is a little more complicated. The front and rear tires will both leave a broken line of 6 inch paint strips one circumference apart, but these two lines will be overlaid. It will look something like this:



2. (10 pts) Nine counters marked with the digits 1 to 9 are placed on the table. Two players alternately take one counter from the table. The winner is the first player to obtain, amongst his counters, three with the sum of exactly 15. What well-known game is this game analogous to? Explain the analogy.

The analogous game is tic-tac-toe. If you make a magic square with the counters before the game starts, such as this one

6	7	2
1	5	9
8	3	4

then the numbers along each row and each column and each diagonal must sum to 15. This is because $1 + 2 + \cdots + 9 = 45$ and the magic square has 3 rows (columns) so each must have a sum of 15. Now picking three counters whose sum is 15 is equivalent to picking three counter along a row, or a column, or a diagonal. That is the objective is exactly the same as in tic-tac-toe.

BTW, neither player in tic-tac-toe has a winning strategy. The best they can do is force a draw.

3. (10 pts) The post office only has stamps of the denominations of 5 cents and 7 cents. What amounts of postage can you buy? Explain your conclusion. What generalizations can you make for stamp denominations m cents and n cents, where m and n are positive integers?

First, we will make a table:

Value (cents)	Combination
1	impossible
2	impossible
3	impossible
4	impossible
5	$1 \cdot 5$
6	impossible
7	$1 \cdot 7$
8	impossible
9	impossible
10	$2 \cdot 5$
11	impossible
12	$1 \cdot 5 + 1 \cdot 7$
13	impossible
14	$2 \cdot 7$
15	$3 \cdot 5$
16	impossible
17	$2 \cdot 5 + 1 \cdot 7$
18	impossible
19	$1 \cdot 5 + 2 \cdot 7$
20	$4 \cdot 5$
21	$3 \cdot 7$
22	$3 \cdot 5 + 1 \cdot 7$
23	impossible
24	$2 \cdot 5 + 2 \cdot 7$
25	$5 \cdot 5$
26	$1 \cdot 5 + 3 \cdot 7$
27	$4 \cdot 5 + 1 \cdot 7$
28	$4 \cdot 7$
29	$3 \cdot 5 + 2 \cdot 7$
30	$6 \cdot 5$
31	$2 \cdot 5 + 3 \cdot 7$
32	$5 \cdot 5 + 1 \cdot 7$
33	$1 \cdot 5 + 4 \cdot 7$

That values like 9 are not possible combinations is easy to verify by subtracting multiples of 7 and checking that the result is never divisible by 5.

It looks like starting with 24, every number is a possible combination. In fact, we can prove this. The table shows how 24, 25, 26, 27, and 28 can be obtained. If $n > 28$, then

$$n - 28, n - 27, n - 26, n - 25, n - 24$$

are five consecutive positive numbers, so one of them must be a multiple of 5. This shows that n can be obtained by starting with a combination that gives 24, 25, 26, 27, or 28 and adding an appropriate number of 5-cent stamps.

What if we have stamps of m and n cents? It would be reasonable to conjecture that as we construct a similar table with those, eventually there will be enough consecutive values that can be made out of these stamps that every value after that will be possible. In fact, we saw in that we need $\min(m, n)$ such consecutive values. But this conjecture would be wrong. For example, if we have 4-cent stamps and 6-cent stamps, we will never be able to make an odd value. Since 2 is divisor of both 4 and 6, 2 will also divide any combination of 4s and 6s. In fact, in general, the greatest common divisor of m and n will divide any combination of m and n-cent stamps. Values that are not divisible by $\gcd(m, n)$ will not be possible combinations.

On the other hand, it is reasonable to conjecture that beyond a certain value, every value which is divisible by $\gcd(m, n)$ will be possible to make. This can indeed be proved, but the proof requires math that is beyond the scope of this class.

4. (10 pts) A very heavy armchair needs to be moved, but the only possible movement is to rotate it through 90° about any of its corners. Can it be moved so that it is exactly beside its starting position and facing the same way?

As we noticed in class, every time we move the chair to an adjacent chair, its orientation changes. If it was facing up or down, it will no face left or right, and vice versa. This suggests that the possible orientation of the chair forms a checkerboard pattern, like this:

↑↓	⇄	↑↓	⇄	↑↓
⇄	↑↓	⇄	↑↓	⇄
↑↓	⇄	↑↓	⇄	↑↓
⇄	↑↓	⇄	↑↓	⇄
↑↓	⇄	↑↓	⇄	↑↓

But how do we know that there is not a longer, more complicated path the chair could take from one square to the next which would return it to the same orientation it started with? If the checkerboard pattern indeed holds, then this is not possible to do. So we need to prove that the checkerboard pattern holds.

Let's label the starting position of the chair by $(0,0)$ and every other square by (x,y) where x is its horizontal coordinate and y is its vertical coordinate:

(−2,0)	(−1,0)	(0,0)	(1,0)	(2,0)
(−2,−1)	(−1,−1)	(0,−1)	(1,−1)	(2,−1)
(−2,−2)	(−1,−2)	(0,−2)	(1,−2)	(2,−2)
(−2,−3)	(−1,−3)	(0,−3)	(1,−3)	(2,−3)
(−2,−4)	(−1,−4)	(0,−4)	(1,−4)	(2,−4)

Notice that each time the chair is moved, either its x-coordinate changes (increases or decreases) by exactly 1 or its y-coordinate changes by exactly 1. Also its orientation is toggled between up/down and left/right. If we start with the up orientation, it will take an even number of moves to end with the up orientation again. But an even number of moves will either change both the x and the y-coordinates by even amounts, or both by odd amounts. This is because you can write an even number as a sum of two even numbers, or as a sum of two odd numbers, but not as a sum of an even and an odd number. But to go from $(0,0)$ to either $(1,0)$ or $(-1,0)$ requires an odd change to the x-coordinate and an even change to the y-coordinate. That is it is not possible to get these positions after an even number of moves. So the chair cannot be moved as requested in the problem.

5. (10 pts) List the typical phases of mathematical problem solving and explain what you would be doing in each phase. Illustrate the phases and what you do in each phase on one of the exercises from this exam.

You can find these on pp. 28, 47, 62, 83, 104, 114, or 131 in Thinking Mathematically. They are

Entry: Read the question carefully, perhaps a few times, reformulate it in your own words, understand what needs to be done, prepare for the attack by introducing appropriate notation.

Attack: Start solving the problem. Experiment with concrete cases, specialize, observe patterns. Generalize and develop a conjecture and justify it.

Review: Check that your work is correct and answers the question. Identify the key ideas in your argument, reflect on what you have learned and what you could have done more efficiently. Extend your results to a more general setting.

I will illustrate these on the postage stamp problem:

Entry: I read and understood the question. I thought about a few examples that were possible to make from the stamps, such as 10, 12, 14 and a few that were not possible, such as 8 and 9. I realized that I should start by constructing a table.

Attack: I constructed the table in my solution. I realized that I could check whether a specific value n could be made from these stamps by subtracting 7, 14, 21, etc, and checking if the result is divisible by 5. Finally, when I got to 29-33, I conjectured that I could now obtain any number higher than these. I informally justified my conjecture by noting that I could always start with 24, 25, 26, 27, or 28 and keep adding 5s to these to obtain all higher values. I formalized my argument by noticing that if I subtract 24, 25, 26, 27, or 28 from n, one of the five numbers I get must be divisible by 5.

I thought about how my result would extend to stamps of denomination m and n. I initially conjectured that it would still be true that after a certain value, all values would be possible to obtain. But I quickly rejected this conjecture when I tried it with $m = 3$ and $n = 6$. I realized that in such a case the value any combination of stamps would have to be divisible by the common denominators of m and n, including the $\gcd(m, n)$. Finally, I realized that if m and n were relatively prime, the same argument that I used for the 5 and 7-cent stamps would likely work, although it may be hard to say at what value I would find $\min(m, n)$ consecutive values that would work. This may be hard to figure out and I would have to think more about it. But I could come up with an argument, why such consecutive values must eventually exist. I also realized that if $d = \gcd(m, n) > 1$, then I could just divide m and n by d, use the same argument for m/d and n/d, which are relatively prime and show that beyond some value, all numbers divisible by d could be obtained as stamp combinations.

Review: I reread my argument and found it convincing. I identified a few key ideas, such as finding 5 consecutive numbers, divisibility by 5, and the fact that 5 and 7 are relatively prime. Finally, I reflected on how I could extend my technique to m and n-cent stamps, although I would have to think more about how soon I will find $\min(m, n)$ consecutive values that worked.

6. (10 pts) List the three main traits of algebraic thinking. Give an example for each from one of the exercises on this exam, or if a trait does not appear on this exam, say so. (Your examples can come from several different exercises.)

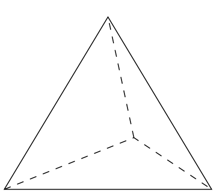
You can find these on pp. 1-2 and 15 in Fostering Algebraic Thinking. They are

Doing-undoing: I did this when solving the armchair problem. I noticed that with each move of the chair, its orientation (up/down versus left/right) would change, and I conjectured that the possible orientations would form a checkerboard pattern. Then I reversed this thinking: I noticed that every even number moves the chair would be in the same orientation as when we began, and reasoned that there was no way to get from the original position to the adjacent position in an even number of moves.

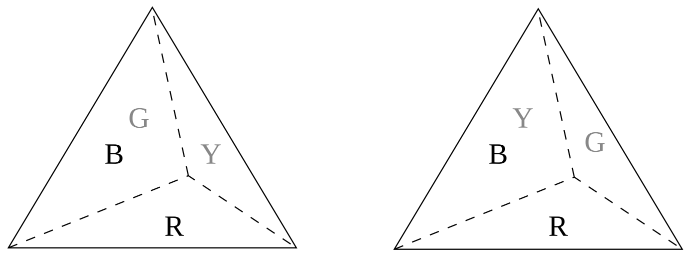
Building functions: Also in the armchair problem, the checkerboard I built is a function: it assigns to every possible position of the chair an orientation.

Abstracting from computation: In the stamps problem, I developed an argument by constructing the table with specific combinations of 5 and 7-cent stamps. Then I realized that I could modify that argument to work with m and n-cent stamps. Even though I cannot really compute with m and n, the key ideas I observed in the specific case carry over to the general case.

7. (10 pts) **Extra credit problem.** The faces of a regular tetrahedron are painted red, blue, green, and yellow, so each face has a different color. In how many different ways can this be done?

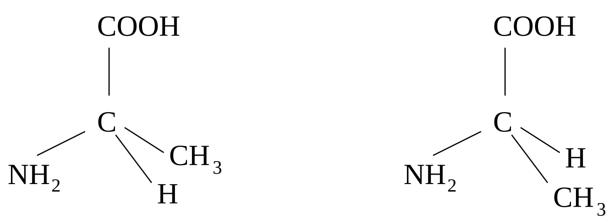


On first approach, we could choose one of four colors for the first face, one of three colors for the second face, one of two colors for the third face, and use the remaining color for the fourth face. This gives 24 colorings. But many of these will actually be the same because the tetrahedron can be rotated in many different ways. In fact, I can always rotate it so that the red face is at the bottom, and the blue face is facing me. Then the green and yellow faces are in the back. Either the green is on the left and the yellow is on the right, or the yellow is on the left and the green is on the right:



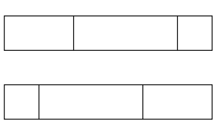
Those two colorings are mirror images of each other. No matter how I rotate the tetrahedron, I cannot rotate one into the other. Just like your left and right hands: they are mirror images of each other, and no matter how you move them, they will not look identical. This is because reflection is orientation-reversing and rotation is orientation-preserving. So there are really two different colorings, shown above.

For your general scientific education: The fact that there are two different ways to color the faces of the tetrahedron and these are mirror images is an important fact in chemistry and biochemistry. It is called chirality. Carbon, which is an essential ingredient of every organic molecule, is usually bound to four other atoms in a tetrahedral arrangement. When those four other atoms (or groups of atoms) are all different, there are two different ways for their orientation about the carbon atom. E.g.



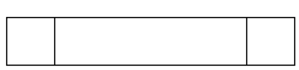
(S)−alanine (R)−alanine

The two resulting molecules are mirror images. They have analogous physical and chemical properties. But biological systems are asymmetric, so the biological activity of the two mirror images would be different. The molecule on the left above is an essential building block of your proteins. The one on the right, does occur in nature in bacteria, but the human body cannot use it.

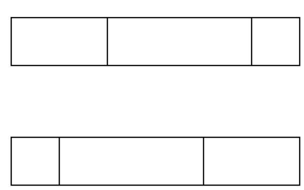


8. (10 pts) **Extra credit problem.** How many Cuisenaire trains of total length n can you make, if rotating the trains about their center point by a 180° is allowed? In other words, the two trains on the left are considered the equivalent.

After a little experimentation, you will realize that there are two kinds of trains: symmetric and asymmetric ones. E.g.



symmetric train



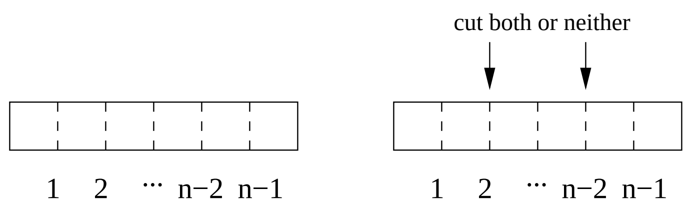
asymmetric pair

The asymmetric ones come in pairs. When we counted them before, we distinguished between an asymmetric train and its rotated counterpart and counted them as two trains. Now that rotation is allowed, we are to count such pairs as one train. This will reduce the number of different trains by exactly the number of such asymmetric pairs. E.g. we used to have 8 different trains of length 4. Now we have 6: there are 4 symmetric ones and 2 pairs of asymmetric ones. You can experiment a little and find the following table:

Length	Symmetric trains	Asymmetric pairs	Total
1	1	0	1
2	2	0	2
3	2	1	3
4	4	2	6
5	4	6	10
6	8	12	20

How do we generalize this? First, we know that there are 2^{n-1} trains of length n if we distinguish head from tail. We also know that there are 2^{n-1} of these because we can start with a bar of length n and choose to cut it or not in $n-1$ places. This gives $n-1$ choices (cut or not) and leads to 2^{n-1} different outcomes.

We can use the same idea to count how many of these are symmetric with respect to rotation. We can still start with a bar of length n, and we still have $n-1$ places where we can cut, but now these cuts are not independent. If I cut at the first place, I also need to cut at the $(n-1)$-st place to maintain symmetry. So the cuts come in pairs, except possibly the middle one, if $n-1$ is odd.



So if $n-1$ is even, we will have $(n-1)/2$ choices, leading to $2^{(n-1)/2}$ different symmetric trains. If $n-1$ is odd, we will have $(n-1+1)/2 = n/2$ choices, leading to $2^{n/2}$ different symmetric trains.

The remaining trains are asymmetric and come in pairs. There are $2^{n-1} - 2^{(n-1)/2}$ or $2^{n-1} - 2^{n/2}$ depending on whether n is odd or even. There are half as many pairs. And that is the number we need to subtract from 2^{n-1}. So if n is odd, we get

$$\begin{aligned} 2^{n-1} - \frac{1}{2}(2^{n-1} - 2^{(n-1)/2}) &= 2^{n-1} - (2^{n-2} - 2^{(n-1)/2-1}) \\ &= 2^{n-1} - 2^{n-2} + 2^{(n-3)/2} \\ &= 2^{n-2} + 2^{(n-3)/2} \end{aligned}$$

different trains. If n is even, we get

$$\begin{aligned} 2^{n-1} - \frac{1}{2}(2^{n-1} - 2^{n/2}) &= 2^{n-1} - (2^{n-2} - 2^{n/2-1}) \\ &= 2^{n-1} - 2^{n-2} + 2^{n/2-1} \\ &= 2^{n-2} + 2^{n/2-1} \end{aligned}$$

different trains.