Math 414 Exam 1 Solutions

Oct 1, 2008

1. (15 pts) Generalize the average test-grade problem from your reading as follows.

   (a) Suppose Jane has taken 4 tests and has average score G so far. What score does she need on the 5th test to average H for all five tests?

   I solved the most general problem in part (c) first. By substituting $n = 4$ and $m = 1$ into the general solution, I get that Jane needs $4(H - G) + H = 5H - 4G$ points on the next 5th test.

   (b) Suppose Jane has taken n tests and has average score G so far. What score does she need on the $n + 1$-st test to average H for all $n + 1$ tests?

   I solved the most general problem in part (c) first. By substituting $m = 1$ into the general solution, I get that Jane needs $n(H - G) + H = (n + 1)H - nG$ points on the next $n + 1$-st test.

   (c) Suppose Jane has taken n tests and has average score G so far. What score does she need on the next m tests to average H for all $n + m$ tests?

   Suppose Jane gets x points on each of the next m tests. Since her cumulative score on the first n tests is nG, she now has a total of $nG + mx$ points. Therefore her average is $(nG + mx)/(n + m)$. We need to find x so that this is H.

$$\begin{aligned}
\frac{nG + mx}{n + m} &= H \\
nG + mx &= (n + m)H \\
mx &= nH + mH - nG \\
x &= \frac{n(H - G)}{m} + H
\end{aligned}$$

   So Jane needs $n(H - G)/m + H$ points on each of her next m tests.

2. (15 pts)

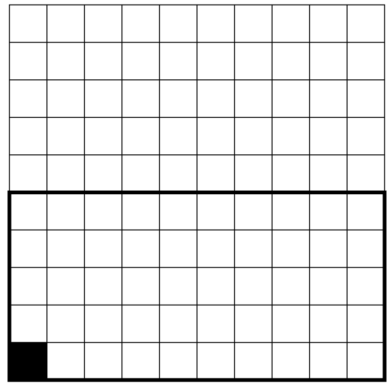
   (a) A solution is 99% water. Some of the water evaporates, leaving a solution that is 98% water. How much (what portion) of the water has evaporated? Solve this problem algebraically.

   We can assume, without loss of generality, that the initial amount of the solution is 1 unit. (We can choose our unit of amount so its equal to the initial volume of the solution.) Then the amount of solute is 0.01 units. This absolute amount remains constant as the water evaporates, but has to become 2% of the solution in the process. Let's say, x units of water evaporate, leaving $1 - x$ units of solution. Then the final concentration of the solute is $0.01/(1 - x)$. Hence

$$\frac{0.01}{1 - x} = 0.02 \implies \frac{1}{2} = 1 - x \implies x = \frac{1}{2}.$$

   That is $1/2$ units of water have to evaporate. We had 0.99 units of water initially, so $0.5/0.99 = 50.\overline{50}\%$ of the water has evaporated.

   (b) This time, use a diagram to solve the problem in part (a).



The diagram above has 99 white squares representing the water and 1 black square representing the solute. The framed area shows that if you remove 50 of the 99 white squares, the black square is exactly 2% of the remaining 50 squares. So 50/99 part of the water needs to evaporate.

(c) A solution is $x\%$ water. Some of the water evaporates, leaving a solution that is $y\%$ water. How much (what portion) of the water has evaporated? (You may assume $y < x$ if you need to.)

We can assume, without loss of generality, that the initial amount of the solution is 1 unit. (We can choose our unit of amount so its equal to the initial volume of the solution.) Then the amount of solute is $(100 - x)/100$ units. This absolute amount remains constant as the water evaporates, but has to become $(100 - y)\%$ of the solution in the process. Let's say, z units of water evaporate, leaving $1 - z$ units of solution. Then the final concentration of the solute is

$$\frac{\frac{100-x}{100}}{1-z} = \frac{100-y}{100}$$

$$\frac{100-x}{100-y} = 1 - z$$

$$z = 1 - \frac{100-x}{100-y} = \frac{(100-y)-(100-x)}{100-y} = \frac{x-y}{100-y}$$

So $(x - y)/(100 - y)$ units of water have to evaporate. We had $x/100$ units of water initially, so the part of the water which has to evaporate is

$$\frac{\frac{x-y}{100-y}}{\frac{x}{100}} = \frac{100(x-y)}{x(100-y)}.$$

3. (15 pts) You have come across the concept of absolute value in your reading and on your homework. The absolute value can be used to define the distance of two real numbers x and y as $|x - y|$. This is an example of a metric:

Definition. A *metric* on a set S is a function $d : S \times S \to \mathbb{R}$ such that
(a) $d(x, y) \geq 0$ for all $x, y \in S$, with equality if and only if $x = y$.
(b) $d(x, y) = d(y, x)$ for all $x, y \in S$.
(c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in S$.

The last of these properties is called the *triangle inequality* because it says that if x, y, z are three points then the sum of two sides of the triangle formed by them is at least as long as the third side.

Prove that the function $d(x, y) = |x - y|$ from $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies this definition. (Hint: proving the triangle inequality requires looking at a few different cases depending on the signs of $x - z$ and $z - y$.)

That $d(x, y) = |x-y| \geq 0$ is obvious from the definition of the absolute value. If $d(x, y) = 0$, then $|x - y| = 0$, but the only number whose absolute value is 0 is 0 itself. So in this case, $x - y = 0$, that is $x = y$.

Since $x - y = -(y - x)$,

$$d(x, y) = |x - y| = |-(y - x)| = |y - x| = d(y, x)$$

for all $x, y \in \mathbb{R}$.

To attack the triangle inequality, we will first show that $|a+b| \leq |a|+|b|$ for any $a, b \in R$. Consider the following cases

$a, b \geq 0$: So $|a| = a$, $|b| = b$ and $a + b \geq 0$. Hence

$$|a + b| = a + b = |a| + |b|$$

which shows that $|a + b| \leq |a| + |b|$ indeed holds.

$a, b < 0$: So $|a| = -a$, $|b| = -b$ and $a + b < 0$. Hence

$$|a + b| = -(a + b) = -a - b = |a| + |b|$$

which shows that $|a + b| \leq |a| + |b|$ indeed holds.

$a \geq 0$, $b < 0$: So $|a| = a$, $|b| = -b$ and $b \leq a + b < a$. So $a + b$ sits between b and a on the real line. Hence its distance from 0 is no more than $\max(|a|, |b|)$. But $|a|$ and $|b|$ are both nonnegative, so

$$|a + b| \leq \max(|a|, |b|) \leq |a| + |b|.$$

$a < 0$, $b \geq 0$: This is just like the previous case with a and b switched (they play a symmetric role anyway). So $|a + b| \leq |a| + |b|$ will still hold.

Now, substitute $a = x - z$ and $b = z - y$ to get

$$d(x, y) = |x - y| = |(x - z) + (z - y)| \leq |x - z| + |z - y| = d(x, z) + d(z, y).$$

4. (15 pts) You have seen analogies between addition and multiplication in class and in your reading. In this exercise, I will list some properties of addition on the additive group $\mathbb{R}$ and will ask you to investigate its analogies on the multiplicative groups $\mathbb{R}^+$ and $\mathbb{R}^*$. ($\mathbb{R}^+$ means the set of positive real numbers, $\mathbb{R}^*$ means the set of nonzero real numbers. Convince yourself that both of these are groups under multiplication.)

   (a) Let $x, y, z \in \mathbb{R}$. If $x + y = x + z$ then $y = z$. Find the analogous statement in terms of multiplication. Does the analogy hold on $\mathbb{R}^+$? Does it hold on $\mathbb{R}^*$?

   We get the analogous statement by replacing addition with multiplication: if $xy = xz$ then $y = z$.
   You can get $y = z$ from $xy = xz$ by dividing by x as long as $x \neq 0$. Since neither $\mathbb{R}^+$ nor $\mathbb{R}^*$ includes 0, the analogous statement holds on both.

   (b) Let $x \in \mathbb{R}$. Then $-(-x) = x$. Find the analogous statement in terms of multiplication. Does the analogy hold on $\mathbb{R}^+$? Does it hold on $\mathbb{R}^*$?

   This statement says that the additive inverse of the additive inverse of x is x. So the analogous statement will say that the multiplicative inverse of the multiplicative inverse of x is x. That is $(x^{-1})^{-1} = x$ for all x.
   A standard property of exponents is that $(x^{-1})^{-1} = x^{(-1)(-1)} = x^1 = x$ for all x. The only problem would be $x = 0$, as 0^{-1} is undefined. But neither $\mathbb{R}^+$ nor $\mathbb{R}^*$ includes 0, the analogous statement holds on both.

(c) Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$. If $x < y$ then $nx < ny$. Find the analogous statement in terms of multiplication. Does the analogy hold on $\mathbb{R}^+$? Does it hold on $\mathbb{R}^*$?

Since n is a positive integer

$$nx = \underbrace{x + x + \cdots + x}_{n \text{ times}}.$$

We will get the analogous statement by replacing each of these additions by a multiplication. This gives

$$\underbrace{xx \cdots x}_{n \text{ times}} = x^n.$$

So the analogous statement is if $x < y$ then $x^n < y^n$.
If $x, y > 0$, this is certainly true, as any power function with positive exponent is increasing over the positive x-axis. So the analogous statement holds on $\mathbb{R}^+$. But it does not hold on $\mathbb{R}^*$ as the following example shows: $-2 < -1$ but $(-2)^2 > (-1)^2$.

5. (15 pts) **Extra credit problem.** Consider n-dimensional Euclidean space R^n. For $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ define

$$d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|, \ldots |x_n - y_n|\}.$$

(a) Prove that d is a metric (see definition in problem 3) on $\mathbb{R}^n$.

Since we are taking the largest of nonnegative numbers, that will also be nonnegative. So $d(x, y) \geq 0$. Suppose $d(x, y) = 0$. Then

$$\max\{|x_1 - y_1|, |x_2 - y_2|, \ldots |x_n - y_n|\} = 0$$

The only way the largest in a set of nonnegative numbers can be 0 is if all the numbers are 0. Hence $x_i - y_i = 0$ for all i. So $x_i = y_i$ for all i, which shows $x = y$.
As in the solution to problem 3, $|x_i - y_i| = |y_i - x_i|$ for all i. Hence

$$d(x, y) = \max\{|x_1 - y_1|, \ldots |x_n - y_n|\} = \max\{|y_1 - x_1|, \ldots |y_n - x_n|\} = d(y, x).$$

Finally, note that by the triangle inequality in problem 3,

$$|x_i - y_i| \leq |x_i - z_i| + |z_i - y_i|.$$

But

$$\begin{aligned} |x_i - z_i| &\leq \max\{|x_1 - z_1|, \ldots |x_n - z_n|\} = d(x, z) \\ |z_i - y_i| &\leq \max\{|z_1 - y_1|, \ldots |z_n - y_n|\} = d(z, y). \end{aligned}$$

So

$$|x_i - y_i| \leq d(x, z) + d(z, y).$$

As this is true for all i, even the largest of $|y_1 - x_1|, \ldots |y_n - x_n|$ cannot be more than $d(x, z) + d(z, y)$. We conclude

$$d(x, y) \leq d(x, z) + d(z, y).$$

(b) Explain why d is called the square metric. (Hint: look at a unit circle.)

Note that the unit circle with respect to this metric is

$$\begin{aligned} \{x \in \mathbb{R}^n \mid d(x, 0) \leq 1\} &= \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid \max\{|x_1|, |x_2|, \ldots, |x_n|\} \leq 1\} \\ &= \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid |x_1|, |x_2|, \ldots, |x_n| \leq 1\} \end{aligned}$$

which describes a square in $\mathbb{R}^2$, a cube in $\mathbb{R}^3$, or a hypercube in $\mathbb{R}^n$ $(n > 3)$. Hence the name.