Math 414 Exam 2 Solutions

Oct 29, 2008

1. (10 pts) Write $\sqrt{3+\sqrt{7}-\sqrt{8-2\sqrt{7}}}$ as the quotient of two integers.

   Note that
   $$8-2\sqrt{7}=1-2\sqrt{7}+7=(1-\sqrt{7})^2.$$
   Hence
   $$\sqrt{8-2\sqrt{7}}=\sqrt{(1-\sqrt{7})^2}=|1-\sqrt{7}|=\sqrt{7}-1.$$
   (Why do we need the absolute value?) So
   $$\sqrt{3+\sqrt{7}-\sqrt{8-2\sqrt{7}}}=\sqrt{3+\sqrt{7}-(\sqrt{7}-1)}=\sqrt{4}=2=\frac{2}{1}.$$

   Remark: Obviously, doing the computation on your calculator and getting 2 does not constitute a valid mathematical argument. Calculators are in general not very useful for proving that a number is rational, since they represent all numbers as finite decimals. To a calculator, any number is rational, hence a quotient of two integers. If you believe your calculator can do such proofs, try using it to prove that $\sqrt{2}$ is a quotient of two integers.

2. (10 pts) Let $n>1$ be an integer and p a prime. Show that $\sqrt[n]{p}$ is irrational. (Hint: Modify the standard proof that $\sqrt{2}$ is irrational.)

   Suppose $s,t\in\mathbb{Z}^+$ such that $\sqrt[n]{p}=s/t$ and s and t are relatively prime, i.e. the fraction is fully reduced. Then
   $$\frac{s^n}{t^n}=p \implies s^n=pt^n$$
   The RHS is divisible by p, so the LHS has to be divisible by p. Since p is prime, this means $p|s$. Hence $s=pk$ for some $k\in\mathbb{Z}$. Then
   $$(pk)^n=pt^n \implies p^{n-1}k^n=t^n$$
   Since $n-1\geq 1$, the LHS is divisible by p. So the RHS is divisible by p. Hence $p|t$. But s and t were supposed to be relatively prime, so they can't have p as a common factor. This contradiction proves that our initial assumption must have been false and $\sqrt[n]{p}$ cannot be written as a quotient of two integers.

3. (10 pts) Consider those *reciprocals of primes* that have *simple-periodic* decimal representations. Using the theorems we proved in class (and in your textbook), prove that of these
   (a) There is exactly 1 with period 1. What is it?

   We are looking for a number p such that $1/p=0.\overline{d}$ where d is some digit. One of the theorems we proved in class says that such a number $0.\overline{d}$ is of the form $\frac{d}{10^1-1}=\frac{d}{9}$. So
   $$\frac{1}{p}=\frac{d}{9} \implies 9=dp$$
   Since the only prime factor of 9 is 3, $p=3$ and hence $d=3$. Indeed, $\frac{1}{3}=0.\overline{3}$ has period 1.

(b) There is exactly 1 with period 2. What is it?

We are looking for a number p such that $1/p = 0.\overline{d_1 d_2}$ where d_1, d_2 are some digits. Using the same theorem as in part (a), such a number $0.\overline{d_1 d_2}$ is of the form $\frac{d_1 d_2}{10^2 - 1} = \frac{d_1 d_2}{99}$. So

$$\frac{1}{p} = \frac{d_1 d_2}{99} \implies 99 = p(d_1 d_2)$$

Since $99 = 3^2 \cdot 11$, p could be 3 or 11. But we already saw that $\frac{1}{3}$ has period 1, so $p = 11$. Indeed, $\frac{1}{11} = 0.\overline{09}$ has period 2.

Note that the reason $p = 3$ shows up here as a possibility again is that the the exponent of 10 in the denominator need not be the period. If that exponent is n, then the first n digits following the decimal point must repeat, but those first n digits could consist of several periods. E.g. $\frac{1}{3} = 0.\overline{33} = \frac{33}{99}$. This is why we had to check that 11 is indeed a solution to the problem.

4. (10 pts) Suppose $\frac{a}{b}$ is a rational number in lowest terms with $b > 0$. Prove that $\frac{a}{b}$ is represented by a decimal which is either finite or infinite periodic of period at most $b - 1$.

This proof is in your textbook in Section 2.1.2.

5. (20 pts)

(a) The numbers $0.\overline{00123}$ and $0.\overline{00813}$ are infinite periodic decimals. Therefore they must be rational numbers. Write each of them as m/n for some m, n integers and simplify these fractions.

$$\begin{aligned} 0.\overline{00123} &= 123\left(\frac{1}{100000} + \frac{1}{100000^2} + \frac{1}{100000^3} + \cdots\right) \\ &= 123\frac{1}{100000}\frac{1}{1 - \frac{1}{100000}} = 123\frac{1}{100000 - 1} = \frac{123}{99999} = \frac{1}{813}. \end{aligned}$$

Similarly,

$$0.\overline{00813} = \frac{813}{99999} = \frac{1}{123}.$$

(b) Stare at the fractions you got in part (a). What connection do you discover? Explain why this connection holds.

Notice

$$\frac{123}{99999} = \frac{1}{813} \implies 123 \cdot 813 = 99999.$$

We proved a theorem in class that said

$$0.\overline{d_1 d_2 \ldots d_p} = \frac{m}{10^p - 1}$$

where $m = d_1 d_2 \ldots d_p$. If $m | 10^p - 1$, then this last fraction simplifies as $1/n$ where $n = \frac{10^p - 1}{m}$. Of course, then it is also true that

$$\frac{1}{m} = \frac{n}{10^p - 1}$$

and the latter fraction has infinite decimal expansion in which the digits of n keep repeating, padded with enough 0s in front of them to make the repeating part p digits long. This is exactly what happens in part (a).

(c) Explain how you can use what you discovered to construct another example like in part (a). (You might as well construct an example.)

We need to find three positive integers m, n, p such that $mn = 10^p - 1$. Ideally, we want m and n to consist of digits that are not periodic, or both to have less than p digits, so it is really apparent in the reciprocals that it is the digits of m and n that repeat.
Let's try $p = 1$. Then $10^p - 1 = 9 = 3^2$. One way to factor 9 is $9 = 1 \cdot 9$. Indeed

$$0.\overline{9} = 1 = \frac{1}{1}, \qquad 0.\overline{1} = \frac{1}{9}.$$

Of course, writing the reciprocal of 1 as $0.\overline{9}$ is strange, so this example is not ideal.
Let's try $p = 2$. Then $10^p - 1 = 99 = 3^2 \cdot 11$. One way to factor 99 is $99 = 9 \cdot 11$. Indeed

$$0.\overline{09} = \frac{1}{11}, \qquad 0.\overline{11} = \frac{1}{9}.$$

The problem is that $0.\overline{11} = 0.111111\ldots$ and it's difficult to spot 11 as the repeating number in that. So this example is not ideal either. The problem is the same with the factoring $99 = 3 \cdot 33$. (Try it!)
Let's try $p = 3$. Then $10^p - 1 = 999 = 3^3 \cdot 37$. One way to factor 999 is $999 = 27 \cdot 37$. Indeed

$$0.\overline{027} = \frac{1}{37}, \qquad 0.\overline{037} = \frac{1}{27}.$$

Finally, we have a good example. You can try other factorings of 999. You'll find they don't give very good example for the same reason factoring 99 didn't give very good examples.

6. (5 pts) **Extra credit problem.** We proved in class that if $x < y$ are real numbers then there exists a rational number r and an irrational number s such that $x < r < y$ and $x < s < y$.

In particular, if $x < y$ are two rational numbers, then there is an irrational number s such that $x < s < y$. Also, if $a < b$ are two irrational numbers, then there is a rational number r such that $a < r < b$. That is between any two rational numbers, there is at least one irrational and between any two irrational numbers, there is at least one rational. Hence the rational and the irrationals alternate on the real line. This shows there are exactly as many rational as irrationals.

Find the mistake in the above argument and explain why it is a mistake.

The mistake is that the argument is that it fails to explain how being able to insert a rational between any two irrationals and an irrational between any two rationals is enough to construct an alternating sequence of rationals and irrationals. How would you start constructing such a sequence? If the rationals (or the irrationals) could be listed in increasing order on the real line, the argument would indeed give a way to fill in the holes among them with numbers of the opposite kind and the alternation would follow. But such a listing is clearly impossible.

You could instead start with any two rationals and insert an irrational between them. If there is no other irrational between them, we have part of the desired alternating sequence. If there is another irrational between the rationals, then take these two irrationals and find a rational between them. If this is the only rational between them, then we have part of the alternating sequence we want. If there is another, find an irrational between those two rationals. And so on. The problem is that this process never finishes, so it never actually constructs part of the alternating sequence which the argument claims exists.

7. (5 pts each) **Extra credit problem.**

(a) Prove that $\sqrt{5}-\sqrt{3}$ is algebraic by finding a polynomial p with integer coefficients such that $p(\sqrt{5}-\sqrt{3})=0$.

Let $a=\sqrt{5}-\sqrt{3}$. Notice that

$$a^2 = 5-2\sqrt{15}+3 = 8-2\sqrt{15}$$
$$a^4 = (a^2)^2 = (8-2\sqrt{15})^2 = 64-32\sqrt{15}+60 = 124-32\sqrt{15}$$

Therefore a^4-16a^2 will have no $\sqrt{15}$ left in it. Indeed

$$a^4-16a^2 = 124-32\sqrt{15}-16(8-2\sqrt{15}) = -4.$$

So $a^4-16a^2+4=0$, i.e. a is a root of $p(x)=x^4-16x^2+4$.

Alternately, notice that

$$a^2 = 8-2\sqrt{15} \implies 8-a^2 = 2\sqrt{15} \implies (8-a^2)^2 = 60$$
$$\implies 64-16a^2+a^4-60=0 \implies a^4-16a^2+4=0$$

gives the same polynomial.

(b) Prove that $1+\sqrt[3]{2}$ is algebraic by finding a polynomial p with integer coefficients such that $p(1+\sqrt[3]{2})=0$.

Let $a=1+\sqrt[3]{2}$. Notice that

$$a^2 = 1+2\sqrt[3]{2}+\sqrt[3]{2}^2$$
$$a^3 = 1+3\sqrt[3]{2}+3\sqrt[3]{2}^2+2 = 3+3\sqrt[3]{2}+3\sqrt[3]{2}^2$$

Therefore

$$a^3-3a^2 = -3\sqrt[3]{2}$$
$$a^3-3a^2+3a = 3$$
$$a^3-3a^2+3a-3 = 0$$

So a is a root of $p(x)=x^3-3x^2+3x-3$.

Another way to discover this polynomial is to notice that

$$(a-1)^3 = \sqrt[3]{2}^3 = 2 \implies$$
$$(a-1)^3-2 = a^3-3a^2+3a-1-2 = a^3-3a^2+3a-3=0.$$