Math 414 Exam 3 Solutions

Nov 24, 2008

1. (10 pts) Prove that if z_1, z_2, and z_3 are complex numbers on the unit circle such that $z_1 + z_2 + z_3 = 0$, then z_1, z_2, and z_3 are vertices of an equilateral triangle. (Hint: Do this in the special case $z_1 = 1$ first. Then in the general case, divide the equation by z_1.)

First, assume that $z_1 = 1$. Let $z_2 = x_2 + y_2 i$ and $z_3 = x_3 + y_3 i$. Then

$$0 = z_1 + z_2 + z_3 = 1 + x_2 + y_2 i + x_3 + y_3 i = (1 + x_2 + x_3) + (y_2 + y_3)i$$

implies $1 + x_2 + x_3 = 0$ and $y_2 + y_3 = 0$. From the latter, $y_3 = -y_2$. Since z_2 and z_3 are on the unit circle

$$x_2^2 + y_2^2 = x_3^2 + y_3^2 = 1.$$

Substitute $y_3 = -y_2$ to get

$$x_2^2 + y_2^2 = x_3^2 + (-y_2)^2 \implies x_2^2 = x_3^2 \implies x_3 = \pm x_2.$$

If $x_3 = -x_2$, then $1 + x_2 + x_3 = 1 \neq 0$, so we can rule out this case. Hence $x_3 = x_2$. Substitute this into $1 + x_2 + x_3 = 0$ to get

$$0 = 1 + x_2 + x_2 \implies x_2 = -\frac{1}{2}.$$

Hence

$$1 = x_2^2 + y_2^2 = \frac{1}{4} + y_2^2 \implies y_2^2 = \frac{3}{4} \implies y_2 = \pm\frac{\sqrt{3}}{2}$$

It doesn't matter which root we choose, since $y_3 = -y_2$. We get either

$$z_2 = \frac{1 + \sqrt{3}i}{2}, z_3 = \frac{1 - \sqrt{3}i}{2} \quad \text{or} \quad z_2 = \frac{1 - \sqrt{3}i}{2}, z_3 = \frac{1 + \sqrt{3}i}{2}.$$

Either way, notice that

$$\frac{1 + \sqrt{3}i}{2} = e^{\frac{2\pi}{3}i}, \qquad \frac{1 + \sqrt{3}i}{2} = e^{\frac{4\pi}{3}i}.$$

So $1, z_2, z_3$ are on the unit circle at $0°$, $120°$, and $240°$. Hence they are the vertices of an equilateral triangle.

For the general case, let $w_2 = z_2/z_1$ and $w_3 = z_3/z_1$ (we know $z_1 \neq 0$ since it's on the unit circle). Notice that $|w_2| = |z_2|/|z_1| = 1$, so w_2 is also on the unit circle. The same goes for w_3. Now

$$1 + w_2 + w_3 = 1 + \frac{z_2}{z_1} + \frac{z_3}{z_1} = \frac{z_1 + z_2 + z_3}{z_1} = 0$$

As we already showed, this implies w_2 and w_3 are at $120°$ and $240°$. Therefore $z_2 = w_2 z_1$ and $z_3 = w_3 z_1$ are $120°$ and $240°$ away from z_1 on the unit circle. Hence z_1, z_2, and z_3 are again equally spaced on the unit circle and form the vertices of an equilateral triangle.

Note that there are many possible solutions, some slicker and the shorter than the above. I chose this because it's very elementary and does not assume too much familiarity with the complex numbers beyond the basic definitions. If you want a worthy challenge, you may try to come up with a shorter proof, using, for example, polar coordinates.

2. (10 pts) Let A and B be finite sets. Suppose A contains x elements and B contains y elements. How many different functions are there from A to B?

Let $f : A \to B$. If $a \in A$, then $f(a) \in B$, so there are y possible choices for what it can be. This is true for each of the x elements in A. We have y choices for each, so we have y^x

choices for the whole function. Each choice will give a different function since at least one element in A will have a different image.

Another way to look at this problem is to use the defition of function. A function $f : A \to B$ is a subset of $A \times B$ such that each element $a \in A$ appears exactly once as the first coordinate of an element in f. That is

$$f = \{(a_1, b_1), (a_2, b_2), \ldots, (a_x, b_x)\}$$

where $a_1, a_2, \ldots, a_x$ is a list of the elements of A. Each b_i can be any of the y elements of B, so there are y choices for each. This gives y^x choices for $b_1, b_2, \ldots, b_x$. Each of these choices gives a different subset of $A \times B$ and therefore a different function. So there are y^x different functions.

3. (10 pts) Let S be a finite set with n elements. How many binary operations are there on S?

   A binary operation is a function $S \times S \to S$. Since $S \times S$ has n^2 elements, the result of problem 2, says that there are n^{n^2} different functions $S \times S \to S$.

4. (12 pts)
   (a) What is a countable set? State the definition.

   A set S is countable if it is finite or has the same cardinality as $\mathbb{N}$, i.e. there exists a one-to-one correspondence $f : \mathbb{N} \to S$.

   Remark: Instead of $\mathbb{N}$, you can just as well use $\mathbb{Z}$ in the above definition.

   (b) Prove that the open interval $(0, 1)$ is not countable. (Hint: Cantor diagonalization.)

   This proof is in Section 2.1.4 of your textbook.

5. (6 pts each)
   (a) Let $z, w \in \mathbb{C}$. Prove that

$$\overline{z+w} = \overline{z} + \overline{w}$$

   and

$$\overline{zw} = \overline{z}\,\overline{w}.$$

   Let $z = x + yi$ and $w = s + ti$. Then

$$\begin{aligned}\overline{z} + \overline{w} &= \overline{x+yi} + \overline{s+ti} = x - yi + s - ti \\ &= x + s - (y+t)i = \overline{x + s + (y+t)i} = \overline{z+w}.\end{aligned}$$

   Also

$$\overline{z}\,\overline{w} = (\overline{x+yi})(\overline{s+ti}) = (x - yi)(s - ti) = xs - yt - (xt + ys)i$$

$$\overline{zw} = \overline{(x+yi)(s+ti)} = \overline{xs - yt + (xt+ys)i} = xs - yt - (xt + ys)i.$$

   Hence $\overline{zw} = \overline{z}\,\overline{w}$.

   (b) Let $z \in \mathbb{C}$. Use the result from part (a) to show that if $n \in \mathbb{Z}^{\geq 0}$ then

$$\overline{z}^n = \overline{z^n}.$$

   (Hint: induction starting at $n = 0$ is a good way to do this.)

Here is the inductive argument. If $n = 0$, then $\overline{z}^0 = 1$ and $\overline{z^0} = 1$, so the statement holds. Assume that it holds for some $n \in \mathbb{Z}^{\geq 0}$. Then

$$\overline{z}^{n+1} = \overline{z}^n \overline{z} = \overline{z^n}\,\overline{z} = \overline{z^n z} = \overline{z^{n+1}},$$

where the second equality is by the inductive hypothesis and the third is by $\overline{z}\,\overline{w} = \overline{zw}$ proved in part (a).

(c) Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with real cofficients (i.e. $a_0, \ldots a_n \in \mathbb{R}$). Prove that if $z \in \mathbb{C}$ is a root of this polynomial, then so is $\overline{z}$. (Hint: show that $p(\overline{z}) = \overline{p(z)}$.)

Notice that $a_i = \overline{a_i}$ since $a_i \in \mathbb{R}$. So

$$\begin{aligned}
p(\overline{z}) &= a_n \overline{z}^n + a_{n-1}\overline{z}^{n-1} + \cdots + a_1 \overline{z} + a_0 \\
&= \overline{a_n}\,\overline{z^n} + \overline{a_{n-1}}\,\overline{z^{n-1}} + \cdots + \overline{a_1}\,\overline{z} + \overline{a_0} && \text{by } \overline{z}^n = \overline{z^n} \text{ and } a_i = \overline{a_i} \\
&= \overline{a_n z^n} + \overline{a_{n-1} z^{n-1}} + \cdots + \overline{a_1 z} + \overline{a_0} && \text{by } \overline{z}\,\overline{w} = \overline{zw} \\
&= \overline{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0} && \text{by } \overline{z} + \overline{w} = \overline{z + w} \\
&= \overline{p(z)}.
\end{aligned}$$

If z is a root of p then $p(z) = 0$. Hence $p(\overline{z}) = \overline{p(z)} = \overline{0} = 0$. This shows $\overline{z}$ is also a root.

6. **Extra credit problem.**

(a) (10 pts) Let $f, g : S \to T$ and $h : T \to U$. Prove that if h is one-to-one and $h \circ f = h \circ g$ then $f = g$. (Recall that $f = g$ means that f and g have the same domain and codomain and $f(x) = g(x)$ for all x in the common domain of f and g.)

Since f and g already have the same domain and codomain, all we need to show is that $f(x) = g(x)$ for all $x \in S$. Let $x \in S$. We know $h \circ f = h \circ g$, so $h(f(x)) = h(g(x))$. But h is one-to-one, so this implies $f(x) = g(x)$.

(b) (5 pts) Give an example of functions $f, g : S \to T$ and $h : T \to U$ such that $f \neq g$ but $h \circ f = h \circ g$. (Be sure to specify what S, T, and U are.)

Let $S = T = U = \mathbb{R}$, $f(x) = \sin(x)$, $g(x) = \cos(x)$, and $h(x) = 0$. Then $h \circ f(x) = 0$ and $h \circ g(x) = 0$ for all $x \in \mathbb{R}$. But it is clear that $f \neq g$.