Math 414 Final Exam Solutions

Dec 17, 2008

1. (10 pts) Prove that the set $\mathbb{Q}$ of all rational numbers is countably infinite. (Hint: First prove that $\mathbb{Q}^+$ is countable.)

   The proof that $\mathbb{Q}^+$ is countable is in your textbook (pp. 44-45).

   An analogous argument shows that $\mathbb{Q}^-$ is countable. Now that we have one-to-one correspondences $f : \mathbb{N} \to \mathbb{Q}^+$ and $g : \mathbb{N} \to \mathbb{Q}^-$, the listing of the elements of $\mathbb{Q}$ as
   $$0, f(1), g(1), f(2), g(2), \ldots$$
   gives a one-to-one correspondence $\mathbb{N} \to \mathbb{Q}$. Hence $\mathbb{Q}$ is countable.

2. (10 pts) Prove that if z_1, z_2, and z_3 are complex numbers on the unit circle such that $z_1 + z_2 + z_3 = 0$, then z_1, z_2, and z_3 are vertices of an equilateral triangle.

   This problem was on the third exam. See its solution on there.

3. (5 pts each) Suppose a and b are chosen from the given set and $*$ is the indicated operation. Does the equation $a * x = b$ always have a unique solution in the set? If not, give an example of an equation with the operation that does not have a unique solution in S.

   (a) set of even integers, multiplication

   The equation does not always have a solution in the set of even integers. For example, it is obvious that none of the following equations have even integer solutions:
   $$\begin{aligned} 0x &= 2 \\ 2x &= 2 \\ 4x &= 2 \end{aligned}$$

   (b) set of odd integers, multiplication

   The equation does not always have a solution in the set of odd integers. For example, the equation $3x = 1$ has no odd integer solution.

4. (10 pts) We defined taxicab distance on $\mathbb{R}^2$ by the formula
   $$d_T\big((x_1, y_1), (x_2, y_2)\big) = |x_1 - x_2| + |y_1 - y_2|.$$
   Prove that d_T has all of the properties required of a distance function. You may use the fact that the absolute value is a distance on $\mathbb{R}$.

   Nonnegativity: Obviously,
   $$d_T\big((x_1, y_1), (x_2, y_2)\big) = |x_1 - x_2| + |y_1 - y_2| \geq 0$$

   Nondegeneracy: Since both absolute values are nonnegative, the only way
   $$0 = d_T\big((x_1, y_1), (x_2, y_2)\big) = |x_1 - x_2| + |y_1 - y_2|$$
   is if both $|x_1 - x_2| = 0$ and $|y_1 - y_2| = 0$. This happens if and only if $x_1 = x_2$ and $y_1 = y_2$.

   Symmetry: This follows from the symmetry of the absolute value.
   $$\begin{aligned} d_T\big((x_1, y_1), (x_2, y_2)\big) &= |x_1 - x_2| + |y_1 - y_2| \\ &= |x_2 - x_1| + |y_2 - y_1| = d_T\big((x_2, y_2), (x_1, y_1)\big) \end{aligned}$$

Triangle inequality: We will use that the absolute value satisfies the triangle inequality on $\mathbb{R}$. (This was on the first exam.) That is

$$|x - y| \leq |x - z| + |z - y|$$

for any $x, y, z \in \mathbb{R}$. Hence

$$\begin{aligned} d_T\big((x_1, y_1), (x_3, y_3)\big) + d_T\big((x_3, y_3), (x_2, y_2)\big) &= |x_1 - x_3| + |y_1 - y_3| + |x_3 - x_2| + |y_3 - y_2| \\ &= \underbrace{|x_1 - x_3| + |x_3 - x_2|}_{\geq |x_1 - x_2|} + \underbrace{|y_1 - y_3| + |y_3 - y_2|}_{\geq |y_1 - y_2|} \\ &\geq |x_1 - x_2| + |y_1 - y_2| = d_T\big((x_1, y_1), (x_2, y_2)\big) \end{aligned}$$

5. (12 pts)
   (a) What is an algebraic number?

   An algebraic number is a number that is a root of a nonzero polynomial with integer coefficients.

   (b) Suppose that r is a nonzero algebraic number. Prove that $1/r$ is also algebraic.

   Since r is algebraic, there exists a nonzero polynomial

   $$p(x) = a_n x^n + \cdots + a_1 x + a_0$$

   with $a_0, a_1, \ldots, a_n \in \mathbb{Z}$ such that $p(r) = 0$. Hence

   $$\begin{aligned} 0 = \frac{1}{r^n} p(r) &= \frac{1}{r^n}(a_n r^n + \cdots + a_1 r + a_0) \\ &= a_n + a_{n-1}\frac{1}{r} + \cdots + a_1 \frac{1}{r^{n-1}} + a_0 \frac{1}{r^n} \end{aligned}$$

   Therefore $1/r$ is a root of the polynomial

   $$q(x) = a_n + a_{n-1}x + \cdots + a_1 x^{n-1} + a_0 x^n.$$

   The coefficients of q are the same as the coefficients of p, so they are still integers. Since p was nonzero, $a_i \neq 0$ for at least one $i = 1, \ldots, n$. Hence q is a nonzero polynomial and $1/r$ is algebraic.

6. (10 pts) Let $m < n$ be relatively prime positive integers. Prove that m/n has a finite decimal representation $0.d_1 d_2 \ldots d_t$ (with $d_t \neq 0$) if and only if $n = 2^r 5^s$ for some $r, s \in \mathbb{N}$ such that $t = \max(r, s)$.

   This is Theorem 2.6 in your text. Its proof is on p. 36.

7. (6 pts each) Let $f : A \to B$ and $g : B \to C$ be functions.
   (a) Prove that if f and g are both one-to-one then $g \circ f$ is one-to-one.

   We need to prove that if $g \circ f(x) = g \circ f(y)$ for some $x, y \in A$, then $x = y$. So suppose $g \circ f(x) = g \circ f(y)$. Then $g(f(x)) = g(f(y))$. Since g is one-to-one, $f(x) = f(y)$. Since f is one-to-one, $x = y$.

   (b) Prove that if $g \circ f$ is onto then g is onto.

   Suppose $g \circ f$ is onto. We need to prove that for any $c \in C$ there exists some $b \in B$ such that $g(b) = c$. So let $c \in C$. Since $g \circ f$ is onto, there exists some $a \in A$ such that $g \circ f(a) = c$. Let $b = f(a)$. Then $b \in B$ and $g(b) = g(f(a)) = c$.

(c) Give an example of functions f and g such that $g \circ f$ is onto but f is not onto. Be sure to specify the domains and codomains and justify your example.

Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = |x|$ and $g : \mathbb{R} \to \{0\}$ be $g(x) = 0$. Then $g \circ f(x) = 0$. Obviously, $g \circ f$ is onto, for example, $g \circ f(1) = 0$. But f is not onto, for example, there is no $x \in \mathbb{R}$ such that $f(x) = -1$.

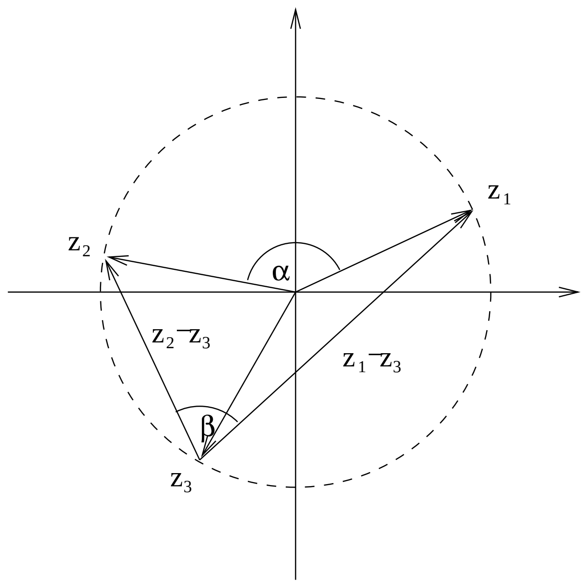
8. **Extra credit problem.** (10 At's) Let $z_1, z_2, z_3 \in \mathbb{C}$ such that $|z_1| = |z_2| = |z_3|$. Prove that

$$\operatorname{Arg}\left(\frac{z_3 - z_2}{z_3 - z_1}\right) = \frac{1}{2}\operatorname{Arg}\left(\frac{z_2}{z_1}\right).$$

(Hint: Draw a picture and remember high school geometry.)

Observe that $|z_1| = |z_2| = |z_3| \neq 0$, otherwise $z_1 = z_2 = z_3 = 0$, and the divisions in the problem do not make sense. So z_1, z_2, z_3 are on a circle.

Remember that when you divide complex numbers in polar coordinates, you divide the absolute values and subtract the angles. That is the argument of the result is the angle between the two complex numbers. Now look at the diagram below.



Notice that

$$\operatorname{Arg}\left(\frac{z_2}{z_1}\right) = \alpha$$

and

$$\operatorname{Arg}\left(\frac{z_3 - z_2}{z_3 - z_1}\right) = \operatorname{Arg}\left(\frac{z_2 - z_3}{z_1 - z_3}\right) = \beta.$$

Notice that α is the central angle of the arc between z_1 and z_2 and β is the inscribed angle of the same arc. We know from high school geometry that the central angle corresponding to an arc is twice the inscribed angle corresponding to the same arc. (This was known to Euclid 2300 years ago!) Therefore $\alpha = 2\beta$. This is what we wanted to prove.