Math 414 Exam 1 Solutions

Feb 21, 2011

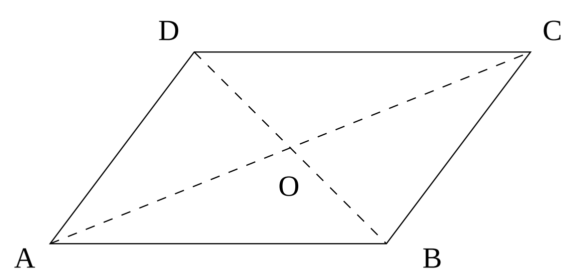
1. (10 pts) A solution is $x\%$ water. Some of the water evaporates, leaving a solution that is $(x-1)\%$ water. Express the amount (portion) of the water that has evaporated as a function of x.

   WLOG, we had a 100 units of solution at the start. Of this, x units were water. Let y be the number of units of water that evaporated. So we now have $100 - y$ units of solution, of which $x - y$ units are water. Hence
   $$\frac{x-y}{100-y} = \frac{x-1}{100}.$$
   We need to solve this equation for y:
   $$\begin{aligned} \frac{x-y}{100-y} &= \frac{x-1}{100} \\ 100(x-y) &= (x-1)(100-y) \\ 100x - 100y &= 100x - xy - 100 + y \\ 100 &= (101-x)y \\ y &= \frac{100}{101-x} \end{aligned}$$
   Since we need the portion of the water that has evaporated, we really want
   $$y/x = \frac{100}{(101-x)x}.$$

2. (10 pts) Suppose "parallelogram" has just been defined as a quadrilateral with two pairs of parallel sides and the following theorem is the first property of parallelograms to be deduced. There is something wrong with the proof of the following theorem. Find the error and correct the proof.

   Theorem. *The diagonals of a parallelogram bisect each other.*

   Proof: Let $ABCD$ be a parallelogram with diagonals $\overline{AC}$ and $\overline{BD}$ intersecting at O. $\overline{AD}$ is parallel to $\overline{BC}$ and $\overline{AB}$ is parallel to $\overline{DC}$ since $ABCD$ is given as a parallelogram. Since alternate angles formed by parallel lines are congruent, $\angle CAD \cong \angle ACB$ and $\angle ADB \cong \angle DBC$. $\overline{AD} \cong \overline{BC}$, since opposite sides of a paralellogram are equal in length. Thus $\triangle AOD \cong \triangle COB$ by ASA congruence. Thus $\overline{AO} \cong \overline{OC}$ and $\overline{BO} \cong \overline{OD}$, since corresponding parts of congruent triangles are congruent. $\square$



   The proof relies on the claim that opposite sides of a parallelogram are congruent. While this is true, it is not obvious from the definition. Since this theorem is the first property of parallelograms, we cannot have an earlier result that says so either. If we want to say opposite sides are congruent, we need to prove it. Here is the corrected version:

Let $ABCD$ be a parallelogram with diagonals $\overline{AC}$ and $\overline{BD}$ intersecting at O. $\overline{AD}$ is parallel to $\overline{BC}$ and $\overline{AB}$ is parallel to $\overline{DC}$ since $ABCD$ is given as a parallelogram. Since alternate angles formed by parallel lines are congruent, $\angle CAD \cong \angle ACB$ and $\angle ACD \cong \angle CAB$. Now $\triangle ACD$ and $\triangle ACB$ share a side and have two pairs of congruent angles, hence they are congruent by ASA. Therefore $\overline{AD} \cong \overline{BC}$ as they are corresponding sides. $\angle ADB$ and $\angle DBC$ are also alternate angles formed by parallel lines, hence they are congruent. Thus $\triangle AOD \cong \triangle COB$ by ASA congruence. Thus $\overline{AO} \cong \overline{OC}$ and $\overline{BO} \cong \overline{OD}$, since corresponding parts of congruent triangles are congruent.

3. (5 pts) Prove that $T(x, y) = (x+5, y)$ is a congruence transformation of the Euclidean plane (with the Euclidean distance).

   Let $(x_1, y_1), (x_1, y_2) \in \mathbb{R}^2$. Then

$$\begin{aligned} d(T(x_1, y_1), T(x_1, y_2)) &= d((x_1+5, y_1), (x_1+5, y_2)) \\ &= \sqrt{((x_1+5)-(x_2+5))^2 + (y_1-y_2)^2} \\ &= \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} \\ &= d((x_1, y_1), (x_1, y_2)) \end{aligned}$$

   Since T preserves distances, T is a congruence transformation.

4. (10 pts) Prove that the following two definitions of least common multiple are equivalent.

   Definition 1. Given two positive integers a and b, the *least common multiple* of a and b is the positive integer m such that
   (a) m is a multiple of both a and b;
   (b) If k is any integer that is a multiple of both a and b, then k is a multiple of m.

   Definition 2. Given two positive integers a and b, the *least common multiple* of a and b is the smallest positive integer m such that m is a multiple of both a and b.

   Let $a, b \in \mathbb{Z}^+$ and let m satisfy Definition 1. We need to show that m also satisfies Definition 2. We already know m is a common multiple of a and b. Suppose k is another common multiple of a and b. By Definition 1, $m|k$, that is $k = qm$ for some $q \in \mathbb{Z}$. Since m and k are both positive, q must be positive. Hence $q \geq 1$. So $k = qm \geq m$. It follows that m is smallest among the positive common multiples of a and b.

   Conversely, let m be a number which satisfies Definition 2. Let k be any other common multiple of a and b. By the Division Algorithm, $k = qm + r$ where $q, r \in \mathbb{Z}$ and $0 \leq r < m$. Since $a|m$ and $a|k$, $a|k - qm = r$. Similarly $b|r$. Since m is the smallest positive common multiple of a and b, and $r < m$, r must not be positive. Hence $r = 0$. So $k = qm$ and $m|k$.

5. (15 pts) Let S be a geometric space with a metric d. Define a *congruence transformation (or isometry)* on S as a one-to-one correspondence $T : S \to S$ such that for all $x, y \in S$, $d(T(x), T(y)) = d(x, y)$. Now define a subset U of S to be *congruent* to a subset V of S if there is a congruence transformation T on S such that $T(U) = V$. Prove that this congruence is an equivalence relation.

   To simplify notation, denote U is congruent to V by $U \cong V$.

   First, notice that the identity map 1_S is a one-to-one correspondence which obviously preserves distances. Hence any $U \subseteq S$ is congruent to itself. That is $\cong$ is reflexive.

Now, let $U, V \subseteq S$ such that $U \cong V$. Hence there exists an isometry T on S such that $T(U) = V$. Since T is a bijection, it has an inverse map T^{-1}, which is also a bijection. Let $x, y \in S$. Let $x' = T^{-1}(x)$ and $y' = T^{-1}(y)$. Since T preserves distances

$$d(T(x'), T(y')) = d(x', y').$$

But $T(x') = T(T^{-1}(x)) = x$ and $T(y') = T(T^{-1}(y)) = y$. Hence

$$d(x, y) = d(T(x'), T(y')) = d(x', y') = d(T^{-1}(x), T^{-1}(y).$$

Therefore T^{-1} also preserves distances. Obviously, $T^{-1}(V) = U$. Hence $V \cong U$. This shows $\cong$ is symmetric.

Finally, let $U, V, W \subseteq S$ such that $U \cong V$ and $V \cong W$. Then there exist isometries T and T' such that $T(U) = V$ and $T'(V) = W$. Since T and T' are both bijections, so is the composite map $T' \circ T$. If $x, y \in S$, then

$$d(x, y) = d(T(x), T(y)) = d(T'(T(x)), T'(T(y)) = d(T' \circ T(x), T' \circ T(y)$$

since T and T' both preserve distances. Finally, notice that $T' \circ T(U) = W$. Hence $U \cong W$. This shows $\cong$ is transitive.

6. (10 pts) **Extra credit problem.** Let T be a congruence transformation of the Euclidean plane $\mathbb{R}^2$ in the sense of the definition of problem 5. Is it true that the image of a circle under T must be a circle? If so, prove it, if not, find a counterexample.

Yes, the image of a circle must be a circle. Let C be a circle with center P and radius r. Then

$$C = \{X \in \mathbb{R}^2 \mid d(P, X) = r\}.$$

Let D be the circle of radius r centered at $T(P)$. We will show $T(C) = D$. First suppose that $Y \in T(C)$. Then there is an $X \in C$ such that $T(X) = Y$. Since T preserves distances

$$d(T(P), Y) = d(T(P), T(X)) = d(P, X) = r.$$

Hence $Y \in D$. It follows that $T(C) \subseteq D$.

Conversely, let $Y \in D$. Since T is onto, there exists a point $X \in \mathbb{R}^2$ such that $T(X) = Y$. Now

$$d(P, X) = d(T(P), T(X)) = d(T(P), Y) = r.$$

This shows $X \in C$. Hence $Y = T(X) \in T(C)$. It follows that $D \subseteq T(C)$. We can now conclude $T(C) = D$.