Math 414 Exam 2 Solutions

Mar 23, 2011

1. (10 pts) Recall that we defined betweenness for points in the plane as B is between A and C if $d(A, B) + d(B, C) = d(A, C)$ where d is the usual Euclidean distance.

   Three distinct points A, B, and C are on a line if and only if one of them is between the other two. Use this statement to prove that the image of a line under isometry is a line.

   Let T be an isometry. First, to simplify notation, if A, B, and C are points, let ABC denote that B is between A and C, that is $d(A, C) = d(A, B) + d(B, C)$. Denote the image of a point X under the transformation T by X'. Since T preserve distance

$$\begin{aligned} ABC &\iff d(A, C) = d(A, B) + d(B, C) \\ &\iff d(A', C') = d(A', B') + d(B', C') \\ &\iff A'B'C'. \end{aligned}$$

   So T also preserves betweenness.

   Let l be a line. We need to prove $T(l)$ is also a line. Let A and B be points on l. Then l is the unique line through A and B. Let $\overleftrightarrow{A'B'}$ be the unique line through A' and B'. We will show that $T(l) = \overleftrightarrow{A'B'}$.

   First, if $X \in l$, $X \neq A, B$ then XAB or AXB or ABX. Hence $X'A'B'$ or $A'X'B'$ or $A'B'X'$. So $X' \in \overleftrightarrow{A'B'}$. Hence we have just shown that $T(l) \subseteq \overleftrightarrow{A'B'}$.

   Now, let $X' \in \overleftrightarrow{A'B'}$. We proved in class that any isometry is onto. Hence there is a point $X \in \mathbb{R}^2$ such that $X' = T(X)$. Since Y is on $\overleftrightarrow{A'B'}$, one of $X'A'B'$, $A'X'B'$, or $A'B'X'$ must be true. But then one of XAB, AXB or ABX must be true too. Hence X is on l. This shows $\overleftrightarrow{A'B'} \subseteq T(l)$. We can now conclude $T(l) = \overleftrightarrow{A'B'}$.

2. (5 pts) Give an analytic proof that a translation in the plane is a congruence transformation.

   Let $T_{(h,k)}$ be a translation along the vector (h, k) and $P = (x_1, y_1)$, $Q = (x_2, y_2)$. Then

$$\begin{aligned} d(T_{(h,k)}(P), T_{(h,k)}(Q)) &= d((x_1 + h, y_1 + k), (x_2 + h, y_2 + k) \\ &= \sqrt{((x_1 + h) - (x_2 + h))^2 + ((y_1 + k) - (y_2 + k))^2} \\ &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= d(P, Q) \end{aligned}$$

3. (10 pts) Find a formula for the image (x', y') of (x, y) under reflection over the line $y = kx$.

   We already know that formula for reflection across a line through the origin at an angle of ϕ to the x-axis:

$$r_\phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & \cos(2\phi) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

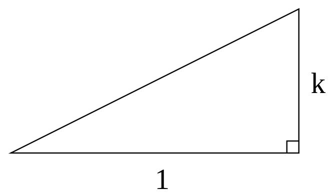
   (BTW, the easy way to remember this formula is to remember it as $r_\phi(z) = e^{2\phi i}\overline{z}$ for $z \in \mathbb{C}$.) So we need to know what ϕ is in terms of k. But that's easy:

$$k = \frac{\Delta y}{\Delta x} = \tan(\phi) \implies \phi = \arctan(k).$$

   Hence

$$r_\phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(2\arctan(k)) & \sin(2\arctan(k)) \\ \sin(2\arctan(k)) & -\cos(2\arctan(k)) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

That works but does not look very elegant. We can clean it up. Notice that in the right triangle



shows

$$\cos(\arctan(k)) = \cos(\phi) = \frac{1}{\sqrt{1+k^2}}$$
$$\sin(\arctan(k)) = \sin(\phi) = \frac{k}{\sqrt{1+k^2}}$$

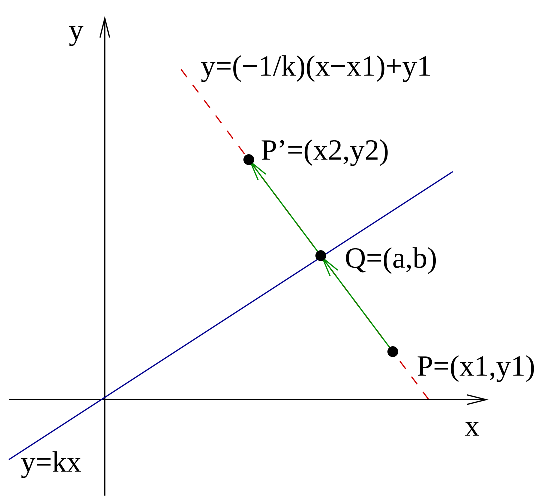
You can verify that this works even if $k < 0$. Hence

$$\cos(2\phi) = \cos^2(\phi) - \sin^2(\phi) = \frac{1}{1+k^2} - \frac{k^2}{1+k^2} = \frac{1-k^2}{1+k^2}$$
$$\sin(2\phi) = 2\sin(\phi)\cos(\phi) = 2\frac{1}{\sqrt{1+k^2}}\frac{k}{\sqrt{1+k^2}} = \frac{2k}{1+k^2}$$

Finally,

$$r_\phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1-k^2}{1+k^2} & \frac{2k}{1+k^2} \\ \frac{2k}{1+k^2} & \frac{k^2-1}{1+k^2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Alternately, you can derive the same formula directly by writing down the equation of a line perpendicular to $y = kx$ through (x_1, y_1), finding the intersection of this line with $y = kx$ (call this (a, b)) and letting $(x_2, y_2) = (x_1, y_1) + 2((a, b) - (x_1, y_1)$ (because $\vec{P'} = \vec{P} + 2\vec{PQ}$). See diagram below.



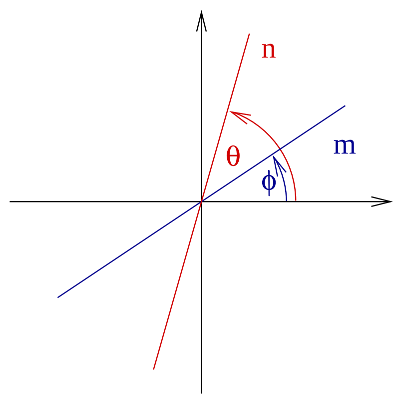
I strongly recommend that you complete this exercise, as it requires no math beyond Algebra 2 and some very elementary vector geometry.

4. (10 pts) Let $R_{C,\phi} : \mathbb{C} \to \mathbb{C}$ be the rotation of the complex plane by ϕ centered at C. Derive a formula for $R_{C,\phi}(z)$.

   See pp. 311-312 in your textbook.

5. (a) (10 pts) Prove the Two-Reflection Theorem for Rotations: If m and n are distinct lines in the plane which intersect at the point P, then $r_n \circ r_m = R_{P,\phi}$ where ϕ is twice the directed angle from m to n.

   See Theorem 7.9(a) in your textbook. We also gave an analytical proof of this in class. Here is a third proof, using complex numbers.
   Without loss of generality, we may assume that P is the origin. This is because there is no special location in the plane that is more of an origin than any other location, and we are in general free to choose what we call the origin. Let m and n make angles of ϕ and θ with the positive real axis respectively.



   Then
   $$r_n \circ r_m(z) = r_n(e^{2\phi i}\overline{z}) = e^{2\theta i}\overline{(e^{2\phi i}\overline{z})} = e^{2\theta i}e^{-2\phi i}z = e^{2(\theta-\phi)i}z = R_{2(\theta-\phi)}(z).$$
   It should be clear from the diagram above that $\theta - \phi$ is in fact the angle from m to n.

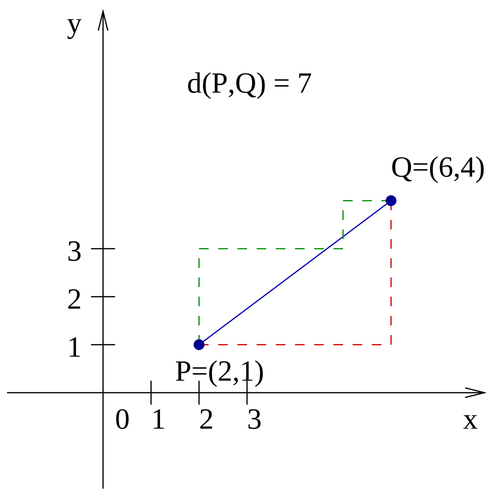
   (b) (5 pts) Find equations of lines m and n such that $r_n \circ r_m$ is the rotation by 60° (in the anticlockwise direction) centered at $(2, 5)$.

   In light of the theorem in part (a), we need two lines which intersect at $(2, 5)$ and have a 30° angle between them. To make life easy, I will choose the first line to be horizontal, that is its slope is 0. Its equation is $y = 5$. The second line will make a 30° angle with the positive x-axis, that is its slope is $\tan(30°) = 1/\sqrt{3}$. Its equation is $y - 5 = 1/\sqrt{3}\,(x - 2)$.

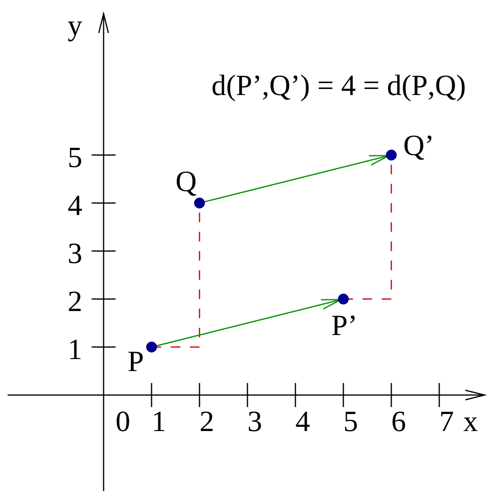
6. **Extra credit problem.** The function $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^{\geq 0}$ given by
   $$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$$
   is called the taxicab distance. (Think about how far a taxi has to drive from one corner to another in a city with regular checkerboard-pattern streets.)

   (a) (6 pts) Is translation an isometry with respect to the taxicab distance? If so, prove it; if not, give a counterexample.

   First, it will help to understand what the taxicab distance really does. The taxicab distance of P and Q is the length of a shortest path from P to Q which consists of horizontal and vertical pieces only. Don't just take my word for this, think about it. There are of course many paths from P to Q that consist of horizontal and vertical pieces, and even the shortest ones among them are not unique. The diagram below shows two such shortest paths.



Now visualize how any translation of the plane will just shift the whole diagram to a new location, without changing the shape of the path or the fact that it is made up of horizontal and vertical pieces.

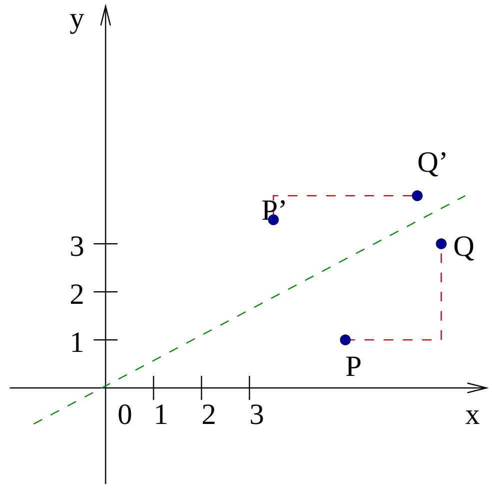


So the image of the path still qualifies for the computation of the taxicab distance from P' to Q'. This is why translations preserve the taxicab distance. Here is an analytic proof with $P = (x_1, y_1)$ and $Q = (x_2, y_2)$:

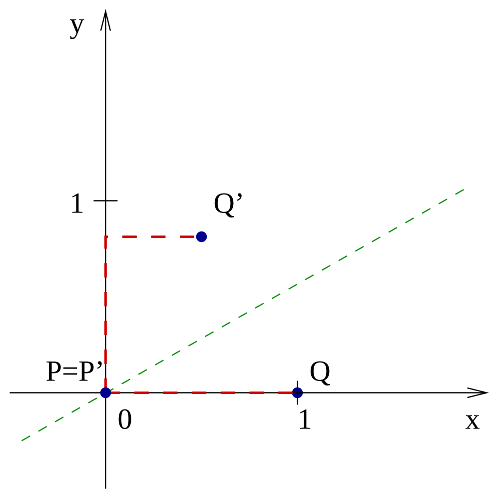
$$\begin{aligned} d(T_{(h,k)}(P), T_{(h,k)}(Q)) &= d((x_1 + h, y_1 + k), (x_2 + h, y_2 + k) \\ &= |(x_1 + h) - (x_2 + h)| + |(y_1 + k) - (y_2 + k)| \\ &= |(x_1 - x_2| + |y_1 - y_2| \\ &= d(P, Q). \end{aligned}$$

(b) (6 pts) Is reflection an isometry with respect to the taxicab distance? If so, prove it; if not, give a counterexample.

Again, let uss first visualize what a reflection might do to one of those shortest paths from P to Q:



It should be quite clear from this picture that there is little reason to expect reflections to preserve the taxicab distance. In fact, it is easy to construct a counterexample. Let $P = (0,0)$ and $Q = (1,0)$. Let r be reflection across a line through the origin at $30°$ from the x-axis:



Then $P' = r(P) = P$ and

$$Q' = r(Q) = (\cos(60°), \sin(60°)) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

So

$$d(P', Q') = \left|0 - \frac{1}{2}\right| + \left|0 - \frac{\sqrt{3}}{2}\right| = \frac{1}{2} + \frac{\sqrt{3}}{2} = \frac{1+\sqrt{3}}{2} \neq 1 = d(P, Q).$$