Math 414 Final Exam Solutions

May 16, 2011

1. (10 pts) In *Mathematique Moderne*, a text for Belgian students written by Georges Papy (1966), a *direction* is defined as a partition of the plane into lines. Then two lines are defined to be *parallel lines* if and only if they are in the same direction. Compare this with the following two definitions of parallel lines:

   Definition 1. Two lines in the same plane are *parallel* if and only if they have no points in common.

   Definition 2. Two lines in the same plane are *parallel* if and only if they are both vertical or they have the same slope.

   Which (if either) of these definitions equivalent to Papy's definition?

   The first definition is not equivalent to Papy's, but the second one is.

   According to Definition 1, a line l is clearly not parallel to itself. But according to Papy's definition, it is. E.g. one direction according to Papy's definition is the partition of the plane into horizontal lines. That this is a partition of the plane is easy to see: every point of the plane is on exactly one horizontal line. The x-axis is in this partition. So the x-axis and the x-axis are in the same partition, and hence they are parallel according to Papy.

   To show that Definition 2 is equivalent to Papy's, we need to show that two lines k and l are parallel according to Definition 2 if and only they are parallel according to Papy's definition. So let k and l be two lines that are parallel by Definition 2. If k and l are both vertical, then they are both in the partition of the plane into vertical lines (this is a partition for the same reason that horizontal lines form a partition). If k and l have the same slope, let this slope be m. Let S_m the set of all lines of slope m in the plane. Then S_m is a partition of the plane because every point is on exactly one line of slope m. So S_m is a direction. Since $k, l \in S_m$, they are parallel by Papy's definition.

   Conversely, suppose k and l are parallel by Papy's definition. Then they are in the same partition of the plane. So either $k = l$ or k and l have no point in common. If $k = l$ then obviously either k and l have the same slope or they are both vertical. So they are parallel by Definition 2. If k and l do not intersect, they cannot possibly have different slopes (lines of different slopes always intersect). So they either have the same slope, or they have no slopes at all, i.e. they are vertical. In both cases, they are parallel according to Definition 2.

2. (10 pts) Prove that every rotation of the Euclidean plane is an isometry (preserves distance) using the formula

   $$R_\phi(x, y) = (x\cos(\phi) - y\sin(\phi), x\sin(\phi) + y\cos(\phi)).$$

   (Hint: You can fill up a whole page with computation, or do this in about four lines if you think before you start computing away.)

   Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$. Then

   $$\begin{aligned}
   &d(R_\phi(P), R_\phi(Q)) \\
   &\quad= \sqrt{((x_1 - x_2)\cos(\phi) - (y_1 - y_2)\sin(\phi))^2 + ((x_1 - x_2)\sin(\phi) + (y_1 - y_2)\cos(\phi))^2} \\
   &\quad= \sqrt{\begin{aligned}&((x_1 - x_2)^2\cos^2(\phi) - 2(x_1 - x_2)\cos(\phi)(y_1 - y_2)\sin(\phi) + (y_1 - y_2)^2\sin^2(\phi) + \\ &(x_1 - x_2)^2\sin^2(\phi) + 2(x_1 - x_2)\sin(\phi)(y_1 - y_2)\cos(\phi) + (y_1 - y_2)^2\cos^2(\phi)\end{aligned}}
   \end{aligned}$$

$$= \sqrt{(x_1 - x_2)^2(\cos^2(\phi) + \sin^2(\phi)) + (y_1 - y_2)^2(\sin^2(\phi) + \cos^2(\phi)}$$
$$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = d(P, Q).$$

3. (10 pts) Let $R_{C,\phi} : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation by (signed) angle ϕ centered at C. Find a formula for $R_{C,\phi}(x, y)$ when $C = (a, b)$.

   We can do this by doing a translation that moves C to the origin, followed by a rotation by ϕ centered at the origin, followed by a translation that moves the origin back to C. That is

$$\begin{aligned} R_{C,\phi}(x, y) &= T_{(a,b)} \circ R_\phi \circ T_{(-a,-b)}(x, y) \\ &= \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \left(\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -a \\ -b \end{pmatrix} \right) + \begin{pmatrix} a \\ b \end{pmatrix} \\ &= ((x - a)\cos(\phi) - (y - b)\sin(\phi) + a, (x - a)\sin(\phi) + (y - b)\cos(\phi) + b). \end{aligned}$$

4. (10 pts) Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be points in R^2. Define the function $d_M : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by
$$d_M(P, Q) = \max(|x_1 - x_2|, |y_1 - y_2|).$$
   Show that d_M is a metric (distance).

   - Non-negativity:
$$0 \le |x_1 - x_2| \le \max(|x_1 - x_2|, |y_1 - y_2|) = d_M(P, Q)$$
   - Non-degeneracy: If
$$0 = d_M(P, Q) = \max(|x_1 - x_2|, |y_1 - y_2|)$$
     then $x_1 - x_2 = 0$ and $y_1 - y_2 = 0$. Hence $P = Q$.
   - Symmetry: Since
$$|x_1 - x_2| = |x_2 - x_1| \text{ and } |y_1 - y_2| = |y_2 - y_1|$$
     we have
$$d_M(P, Q) = \max(|x_1 - x_2|, |y_1 - y_2|) = \max(|x_2 - x_1|, |y_2 - y_1|) = d_M(Q, P)$$
   - Triangle inequality: Let $R = (x_3, y_3)$. Then $d_M(P, Q) = \max(|x_1 - x_3|, |y_1 - y_3|)$. Either
$$\max(|x_1 - x_3|, |y_1 - y_3|) = |x_1 - x_3|$$
     or
$$\max(|x_1 - x_3|, |y_1 - y_3|) = |y_1 - y_3|.$$
     In the first case,
$$\begin{aligned} d_M(P, R) &= |x_1 - x_3| \\ &\le |x_1 - x_2| + |x_2 - x_3| \\ &\le \max(|x_1 - x_2|, |y_1 - y_2|) + \max(|x_2 - x_3|, |y_2 - y_3|) \\ &= d_M(P, Q) + d_M(Q, R). \end{aligned}$$

In the second case,

$$\begin{aligned} d_M(P,R) &= |y_1 - y_3| \\ &\le |y_1 - y_2| + |y_2 - y_3| \\ &\le \max(|x_1 - x_2|, |y_1 - y_2|) + \max(|x_2 - x_3|, |y_2 - y_3|) \\ &= d_M(P,Q) + d_M(Q,R). \end{aligned}$$

5. (10 pts) We proved in class that the composition of any three reflections is either a reflection or a glide reflection. Use this result to show that any composition of reflections, rotations, translations, and glide reflections is itself either a reflection, a rotation, a translation, or a glide reflection.

Reflections, rotations, translations, and glide reflections can all be written as compositions of reflections. So it is enough to prove that any composition of reflections is a reflection, a rotation, a translation, or a glide reflection. We will do this by induction. Let T be a composition of n reflections. The statement is trivially true if $n = 1$. If $n = 2$, it is given by the two-reflection theorems (Theorems 7.9(a) and (b)). The result is given for $n = 3$ in the statement of the problem.

For $n = 4$,

$$T = r_d \circ r_c \circ r_b \circ r_a = r_d \circ (r_c \circ r_b \circ r_a).$$

We know $r_c \circ r_b \circ r_a$ is either a reflection or a glide reflection. If it is a reflection, T is a composition of two reflections, hence we are back in the $n = 2$ case. If $r_c \circ r_b \circ r_a$ is a glide reflection, say $r_e \circ T_{\vec{v}}$, then

$$T = r_d \circ r_e \circ T_{\vec{v}}.$$

If d and e are parallel, then $r_d \circ r_e$ is a translation (by Theorem 7.9(b)). Then T is a composition of two translations, which is a translation, which is a composition of two reflections. If d and e intersect, then rewrite $T_{\vec{v}}$ as $r_g \circ r_f$ where f and g are parallel lines. By the Translation Flexibility Theorem (Theorem 7.10(b)) we can choose f and g so that g is concurrent with d and e. Then

$$T = r_d \circ r_e \circ r_g \circ r_f = \underbrace{(r_d \circ r_e \circ r_g)}_{r_h} \circ r_f,$$

where r_h is a reflection by the proof of the 3-reflection theorem (Theorem 7.15, in the statement of this problem). Hence T is a rotation or a translation. In either case, T is a composition of two reflections.

Now, let $n > 4$ and assume that the statement has been proven for all $k < n$. Take the first four reflections in T. By the argument we have just given, these can be replaced with a composition of two reflections. So T is also a composition of $n - 2$ reflections. By the inductive hypothesis, T is a reflection, a rotation, a translation, or a glide reflection.

6. Our textbook defined a transformation of the plane $\mathbb{R}^2$ as a one-to-one function $T : \mathbb{R}^2 \to \mathbb{R}^2$. We defined an isometry of $\mathbb{R}^2$ as a map $T : \mathbb{R}^2 \to \mathbb{R}^2$ which preserves distance.
   (a) (5 pts) Prove that every isometry is one-to-one.

   Let T be an isometry and $P, Q \in \mathbb{R}^2$. Suppose $T(P) = T(Q)$. Then

   $$0 = d(T(P), T(Q)) = d(P,Q) \implies P = Q.$$

   Hence T is one-to-one.

   (b) (10 pts) Does there exist an isometry of the plane which is not onto? If so, find an example (and justify that it is an example). If not, give an argument why you do not expect to be able to find such an isometry.

No such thing should exist. The reason has to do with triangulation: given three distinct points P, Q, R in $\mathbb{R}^2$, any other point X is uniquely determined by its distances from P, Q and R.

Let T be an isometry of $\mathbb{R}^2$ and choose three distinct points P, Q and R. Let $P' = T(P)$, $Q' = T(Q)$, and $R' = T(R)$. Now, let Y be any point in $\mathbb{R}^2$. Let $a = d(Y, P')$, $b = d(Y, Q')$, and $c = d(Y, R')$. Use a, b, c to locate the unique point X in $\mathbb{R}^2$ whose distances from P, Q and R are a, b, and c respectively. That such a point exists is guaranteed by the fact $\triangle PQR \cong \triangle P'Q'R'$, which is so because T preserves the distances between these points. Now, let $X' = T(X)$. Since

$$\begin{aligned} d(X', P') &= d(X, P) = a = d(Y, P') \\ d(X', Q') &= d(X, Q) = b = d(Y, Q') \\ d(X', R') &= d(X, R) = c = d(Y, R') \end{aligned}$$

X' must be the unique point whose distances from P', Q' and R' are a, b and c. But that point is Y. So $Y = T(X)$. That is every point of $\mathbb{R}^2$ is in the image of T.

(c) (5 pts) Does there exist a one-to-one correspondence $T : \mathbb{R}^2 \to \mathbb{R}^2$ which is not an isometry with respect to the Euclidean distance? If so, find one (and justify that it is an example). If not, argue why such a one-to-one correspondence does not exist.

There are many such. For example, dilation by a factor of 2 centered at the origin $T(x, y) = (2x, 2y)$. This is clearly a one-to-one correspondence since it has an obvious inverse $S(x, y) = (x/2, y/2)$. But it is not an isometry. E.g.

$$d(T(1, 0), T(0, 0)) = d((2, 0), (0, 0)) = 2 \neq 1 = d((1, 0), (0, 0)).$$

7. (a) (10 pts) Let l be a line in the complex plane which passes through the origin and makes a (signed) angle of ϕ with the x-axis. Let $r_l : \mathbb{C} \to \mathbb{C}$ denote reflection across l. Derive a formula for r_l.

See Theorem 7.11.

8. (10 pts each) **Extra credit problem.** Let $d_1, d_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^{\geq 0}$ be two distance functions on the plane. We say d_1 is *equivalent* to d_2 if there exist positive real constants m and n such that

$$m d_1(P, Q) \leq d_2(P, Q) \leq n d_1(P, Q)$$

for all points $P, Q \in \mathbb{R}^2$.

(a) Prove that equivalence of distances is an equivalence relation.

Let d_1, d_2, and d_3 be distances. Denote d_1 is equivalent to d_2 by $d_1 \sim d_2$.

- Reflexivity: Notice that $d_1 \sim d_1$ because $1d_1(P, Q) \leq d_1(P, Q) \leq 1d_1(P, Q)$.
- Symmetry: Suppose $d_1 \sim d_2$. Then

  $$m d_1(P, Q) \leq d_2(P, Q) \leq n d_1(P, Q)$$

  for some $m, n \in \mathbb{R}^+$. Hence

  $$\frac{1}{n} d_2(P, Q) \leq d_1(P, Q) \leq \frac{1}{m} d_2(P, Q).$$

  So $d_2 \sim d_1$.

- Transitivity: Suppose $d_1 \sim d_2$ and $d_2 \sim d_3$. Then
$$md_1(P,Q) \leq d_2(P,Q) \leq nd_1(P,Q)$$
for some $m, n \in \mathbb{R}^+$, and
$$m'd_2(P,Q) \leq d_3(P,Q) \leq n'd_2(P,Q)$$
for some $m', n' \in \mathbb{R}^+$ Hence
$$mm'd_1(P,Q) \leq m'd_2(P,Q) \leq d_3(P,Q) \leq n'd_2(P,Q) \leq nn'd_1(P,Q).$$
Since mm' and nn' must be positive real numbers, $d_1 \sim d_3$.

(b) Prove that the Euclidean distance $d_E((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$ is equivalent to the taxicab distance $d_T((x_1,y_1),(x_2,y_2)) = |x_1-x_2|+|y_1-y_2|$.

First, notice that
$$\begin{aligned} d_E((x_1,y_1),(x_2,y_2))^2 &= (x_1-x_2)^2+(y_1-y_2)^2 \\ &\leq (x_1-x_2)^2+(y_1-y_2)^2 + 2\underbrace{|x_1-x_2||y_1-y_2|}_{\geq 0} \\ &= |x_1-x_2|^2+|y_1-y_2|^2+2|x_1-x_2||y_1-y_2| \\ &= d_T((x_1,y_1),(x_2,y_2))^2 \end{aligned}$$
Since distances are nonnegative numbers and the square root function is increasing on the nonnegative reals, this implies
$$d_E((x_1,y_1),(x_2,y_2)) \leq d_T((x_1,y_1),(x_2,y_2)).$$
Now, notice that
$$\begin{aligned} &2d_E((x_1,y_1),(x_2,y_2))^2 - d_T((x_1,y_1),(x_2,y_2))^2 \\ &= 2(x_1-x_2)^2+2(y_1-y_2)^2-|x_1-x_2|^2-|y_1-y_2|^2-2|x_1-x_2||y_1-y_2| \\ &= (x_1-x_2)^2+(y_1-y_2)^2-2|x_1-x_2||y_1-y_2| \\ &= \big(|x_1-x_2|+|y_1-y_2|\big)^2 \geq 0 \end{aligned}$$
Hence
$$2d_E((x_1,y_1),(x_2,y_2))^2 \geq d_T((x_1,y_1),(x_2,y_2))^2.$$
Once again, the square root function is increasing on the nonnegative reals, so
$$\sqrt{2}d_E((x_1,y_1),(x_2,y_2)) \geq d_T((x_1,y_1),(x_2,y_2)).$$
Therefore
$$1d_E((x_1,y_1),(x_2,y_2)) \leq d_T((x_1,y_1),(x_2,y_2)) \leq \sqrt{2}d_E((x_1,y_1),(x_2,y_2)).$$
So d_E is equivalent to d_T.