Math 510 Exam 1 Solutions

Oct 11, 2010

1. (5 pts each) Define the following terms:
   (a) The *triangle* $\triangle ABC$ formed by three noncollinear points A, B, and C.

   Let A, B, C be three noncollinear points. The *triangle* $\triangle ABC$ is the triple of line segments (AB, BC, CA).
   Alternately, you could define *triangle* $\triangle ABC$ to be the set of line segments $\{AB, BC, CA\}$ or the union of line segments $AB \cup BC \cup CA$.

   (b) The *vertices*, *sides*, and *angles* of $\triangle ABC$. (The sides are segments, not lines.)

   The *vertices* of $\triangle ABC$ are the points A, B, C. The *sides*, and of $\triangle ABC$ are the line segments AB, BC, CA. The *angles* of $\triangle ABC$ are $\sphericalangle BAC, \sphericalangle ABC, \sphericalangle BCA$.

2. (10 pts) Given the axioms of Incidence Geometry, prove that for every point, there is at least one line not passing through it. You may use any other propositions in Incidence Geometry, except Prop 2.4 (because this is Prop 2.4).

   Let P be any point. By Prop 2.2, there exist three nonconcurrent lines l, m, n. At least one of these lines does not pass through P, otherwise they would be concurrent at P.

3. Let A and B be points in some Euclidean geometry.
   (a) (5 pts) Define the line segment AB.

   Given distinct points A and B, the line segment AB is the set

   $$AB = \{A, B\} \cup \{C \mid C \text{ is on the line } \overleftrightarrow{AB} \text{ and } C \text{ is between } A \text{ and } B\}.$$

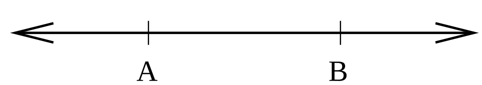
   (b) (5 pts) Define the ray $\overrightarrow{AB}$.

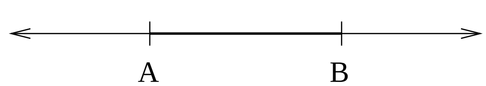
   Given distinct points A and B, the ray $\overrightarrow{AB}$ is the set

   $$\overrightarrow{AB} = \{A, B\} \cup \{C \mid C \text{ is on the line } \overleftrightarrow{AB} \text{ and } C \text{ is between } A \text{ and } B\} \cup$$
   $$\{C \mid C \text{ is on the line } \overleftrightarrow{AB} \text{ and } B \text{ is between } A \text{ and } C\}.$$

4. (10 pts) If S and T are any sets, their *union* $(S \cup T)$ and *intersection* $(S \cap T)$ are defined as follows:
   (a) Something belongs to $S \cup T$ is and only if it belongs to either S or to T (or both of them).
   (b) Something belongs to $S \cap T$ is and only if it belongs to both S and to T.

   Given two points A and B, consider the two rays $\overrightarrow{AB}$ and $\overrightarrow{BA}$. Draw diagrams to show that $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$ and $\overrightarrow{AB} \cap \overrightarrow{BA} = AB$. What additional axioms about the undefined term "between" must we assume in order to be able to prove these equalities?





For $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$, we need

$$\overrightarrow{AB} \cup \overrightarrow{BA} \subseteq \overleftrightarrow{AB} \quad \text{and} \quad \overleftrightarrow{AB} \subseteq \overrightarrow{AB} \cup \overrightarrow{BA}.$$

The very first problem here is that a line is an undefined primitive and we never said it was a set. So equality and inclusion are currently meaningless in reference to lines. So first of all, we need an axiom that says we will use a line interchangeably with the set of all points that lie on that line. So $\overleftrightarrow{AB}$ does not only mean the unique line (an undefined primitive) such that A and B lie on it, but it also means the set

$$\overleftrightarrow{AB} = \{C \mid C \text{ is a point that lies on } \overleftrightarrow{AB}\}.$$

While you may initially find using the same line to mean two different things confusing, in practice it works fine and is usually how we do it in geometry. Now, the first of these inclusions is clear by the definition of ray since every point in $\overrightarrow{AB}$ or $\overrightarrow{BA}$ is required to be a point on the line $\overleftrightarrow{AB}$, hence it belongs to the set $\overleftrightarrow{AB}$. For the reverse inclusion, we would need to know that if C is a point on $\overleftrightarrow{AB}$ other than A and B, then C satisfies at least one of the following:

(a) C is between A and B (so $C \in \overrightarrow{AB}$),
(b) C is between B and A (so $C \in \overrightarrow{BA}$),
(c) B is between A and C (so $C \in \overrightarrow{AB}$),
(d) A is between B and C (so $C \in \overrightarrow{BA}$).

Since between is an undefined primitive, we would need an axiom that states this.

For $\overrightarrow{AB} \cap \overrightarrow{BA} = AB$, we need

$$\overrightarrow{AB} \cap \overrightarrow{BA} \subseteq AB \quad \text{and} \quad AB \subseteq \overrightarrow{AB} \cap \overrightarrow{BA}.$$

The second of these inclusions requires that A, B, and any point on $\overleftrightarrow{AB}$ between A and B is an element of both $\overrightarrow{AB}$ and $\overrightarrow{BA}$. No problem for A and B. They are by definition elements of both $\overrightarrow{AB}$ and $\overrightarrow{BA}$. Now, if C is a point on $\overleftrightarrow{AB}$ that is distinct from A and B and is between A and B then by definition $C \in \overrightarrow{AB}$. But it is not clear that $C \in \overrightarrow{BA}$. For that, C has to be between B and A too. While the use of between in everyday language would suggest that this must be so, we have no definition of between and no axiom about between that would tell us that if C is between A and B then C must also be between B and A. So that is what we need to have as an axiom.

As for the first, any point $C \in \overrightarrow{AB} \cap \overrightarrow{BA}$ has to be in $\overrightarrow{AB}$. If $C = A$ or $C = B$ or C is between A and B then $C \in AB$. So we are good. But if B is between A and C and somehow also $C \in \overrightarrow{BA}$, then it is not clear that C also has to be in AB. One way to address this problem is to have an axiom that for three distinct collinear points A, B, C, if B is between A and C, then A is not between B and C, and C is not between B and A. This way those points $C \in \overrightarrow{AB}$ that are such that B is between A and C are excluded from $\overrightarrow{AB} \cap \overrightarrow{BA}$. The only points that remain in $\overrightarrow{AB} \cap \overrightarrow{BA}$ are A, B and those points C that are between A and B (not necessarily all of them).

5. (5 pts each)

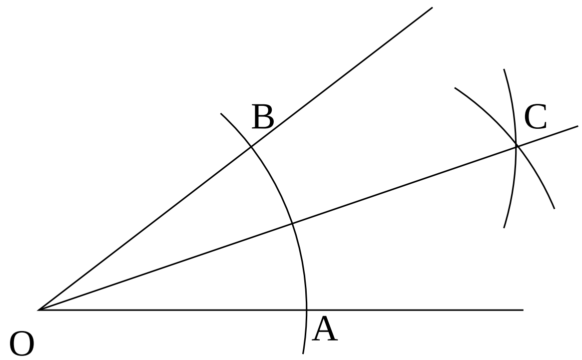
(a) State the dual of Incidence Axiom 1.

For any two distinct lines l and m there exists a unique point which is incident with both l and m.

(b) State the negation of Incidence Axiom 1 without using phrases like "It is not true that" or "It is not the case that."

There exist two points A and B such that either there is no line passing through both A and B or there are at least two distinct lines passing through both A and B.

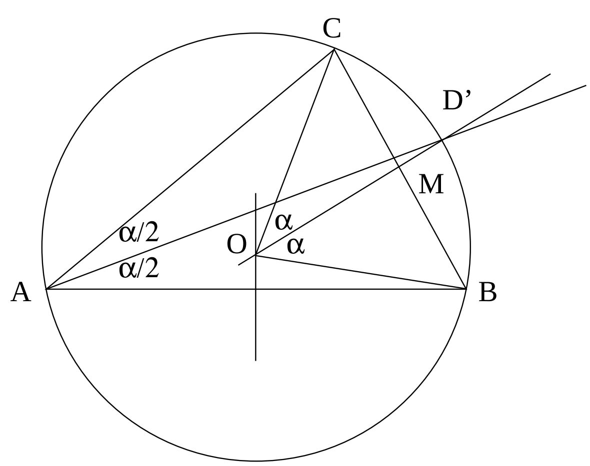
6. (10 pts) Bisect the angle below with straight edge and compass only. Prove that your construction does indeed bisect the angle.



1. Draw a circle centered at the vertex of the angle with any radius. Let the points of intersection of this circle with the sides of the angle be A and B.
2. Draw circles centered at A and B with radius OA. These circles intersect at O, and one other point, call that other point C.
3. Draw the ray $\overrightarrow{OC}$. This ray bisects $\sphericalangle AOB$.

To see that $\sphericalangle AOC \cong \sphericalangle BOC$, notice that $\triangle AOC \cong \triangle BOC$. This is because their corresponding sides are congruent: they share side OC, $OA \cong OB$ and $AC \cong BC$ by construction.

7. (10 pts) **Extra credit problem.** Consider the triangle below, in which AB and AC are not congruent. D is the intersection of the bisector of the angle $\sphericalangle BAC$ and the perpendicular bisector of the side BC. Prove that D lies on the circumcircle of $\triangle ABC$. (The circumcircle of a triangle is the unique circle which passes through all three vertices.)



We will give a slightly indirect proof. Let O be the center of the circumcircle and M the midpoint BC. Let D' be the point where the perpendicular bisector of BC intersects the circumcircle on the same side of BC as D is (see diagram). Notice that $BM = MC$, $OB = OC$, and $OM = OM$, hence $\triangle OMB \cong \triangle OMC$. This shows $\sphericalangle BOM = \sphericalangle COM$. But $\sphericalangle BOM = \sphericalangle BOD'$ which is the central angle corresponding to the arc BD'. $\sphericalangle BAD'$

is an inscribed angle corresponding to arc BD'. By the Central Angle Theorem $2\sphericalangle BAD' = \sphericalangle BOD'$. Analogously, $2\sphericalangle CAD' = \sphericalangle COD'$. Hence

$$2\sphericalangle BAD' = \sphericalangle BOD' = \sphericalangle BOM = \sphericalangle COM = \sphericalangle COD' = 2\sphericalangle CAD',$$

and so $\sphericalangle BAD' = \sphericalangle CAD'$. This shows that D' lies on the bisector of $\sphericalangle BAC$. Therefore $D' = D$. We can now conclude that D lies on the circumcircle.