Math 510 Exam 2 Solutions

Nov 15, 2010

1. (10 pts) In the context of incidence geometry, prove that for every point P, there exist at least two distict lines through P. This is Prop 2.5 in the textbook, so you obviously cannot use Prop 2.5 in your argument.

   Let P be any point. By Axiom I-3, there exist three distinct noncollinear points A, B, C. Now there are two cases:

   P **is one of the three noncollinear points:** Without loss of generality, $P = A$ (otherwise just relabel the points). By Axiom I-1, there is a unique line through A and B and a unique line through A and C. Call these $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$. These two lines must be distinct, otherwise A, B, C would lie on the same line. So $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ are two distinct lines through $A = P$.

   P **is not one of the three noncollinear points:** Now consider the three unique lines $\overleftrightarrow{AP}$, $\overleftrightarrow{BP}$, and $\overleftrightarrow{CP}$ (by Axiom I-1). If these three lines were all equal, then A, B, C would all lie on this line. That would contradict the noncollinearity of A, B, C. So at least two of these lines are distinct.

2. (10 pts) Show that when each of two models of incidence geometry has exactly three points in it, the models are isomorphic.

   Let M_1 and M_2 be models of incidence geometry with 3 points each. Call the points in M_1 A_1, B_1, C_1 and the points in M_2 A_2, B_2, C_2. By Axiom I-3, both models must have 3 distinct noncollinear points. Since each model has exactly 3 distinct points, those 3 points must be noncollinear.

   By Axiom I-1, M_1 must have a unique line through A_1 and B_1. Call this line $\overleftrightarrow{A_1B_1}$. Point C_1 cannot lie on $\overleftrightarrow{A_1B_1}$ by the noncollinearity of A_1, B_1, C_1. Similarly, M_1 must have unique lines $\overleftrightarrow{A_1C_1}$ and $\overleftrightarrow{B_1C_1}$, which do not contain the third point. These lines are obviously distinct (e.g. because each is not incident with a different point). We will show that these are all the lines in M_1. Suppose l is a line in M_1. By Axiom I-2, l must contain at least two points in M_1. It cannot contain 3 points, otherwise A_1, B_1, C_1 would be collinear. So l contains exactly two points, namely A_1 and B_1, or A_1 and C_1, or B_1 and C_1. That is l is one of the lines $\overleftrightarrow{A_1B_1}$, $\overleftrightarrow{A_1C_1}$, or $\overleftrightarrow{B_1C_1}$.

   An analogous argument shows that M_2 must have exactly three distinct lines $\overleftrightarrow{A_2B_2}$, $\overleftrightarrow{A_2C_2}$, and $\overleftrightarrow{B_2C_2}$, and none of these pass through the third point.

   We are now ready to set up a one-to-one correspondence between points and lines in the two models:

$$\begin{array}{rcl} M_1 & & M_2 \\ A_1 & \leftrightarrow & A_2 \\ B_1 & \leftrightarrow & B_2 \\ C_1 & \leftrightarrow & C_2 \\ \overleftrightarrow{A_1B_1} & \leftrightarrow & \overleftrightarrow{A_2B_2} \\ \overleftrightarrow{A_1C_1} & \leftrightarrow & \overleftrightarrow{A_2C_2} \\ \overleftrightarrow{B_1C_1} & \leftrightarrow & \overleftrightarrow{B_2C_2} \end{array}$$

It is quite clear that these correspondences respect incidence. Here is an example. $\overleftrightarrow{A_1B_1} \mathrm{I} A_1, B_1$ but $\overleftrightarrow{A_1B_1} \not\mathrm{I} C_1$. The corresponding line in M_2 is $\overleftrightarrow{A_2B_2}$, and $\overleftrightarrow{A_2B_2} \mathrm{I} A_2, B_2$ but $\overleftrightarrow{A_2B_2} \not\mathrm{I} C_2$.

Note: Many of you mistook this problem for what is done in Example 5 on p. 80. That example shows you that two particular 3-point models of incidence geometry are isomorphic, not that any two 3-point models are isomorphic. Those are two different things, although some parts of the two proofs are similar. If you don't understand what the difference is between this statement and what is proven in Example 5, come and talk to me. It's important that we clear up the difference.

3. (10 pts) Assuming the context of the axiomatic geometry we have been building in class (see axioms at the end of this exam), prove that if A, B, C, and D are points such that ABC and BCD, then A, B, C, D are distinct and collinear. This is Theorem 3(b), so you cannot use Theorem 3 in your proof, but you are allowed to use any of the other axioms and theorems in your argument.

By Axiom 6, A, B, and C are distinct. By the same axiom applied to BCD, B, C, and D are distinct. Notice that A and D could still be the same. Suppose $A = D$. Then BCD is the same as BCA. But ABC and BCA contradict according to Theorem 1. So $A \neq D$, and A, B, C, D are distinct.

By Axiom 6, ABC implies that A, B, and C lie on some line k. Similarly, BCD implies B, C, and D lie on a line l. But k and l both go through the distinct points B and C, hence Axiom 3 shows $k = l$. Therefore A, B, C, D are collinear.

4. Let $\mathcal{P}$ be a projective plane.
   (a) (3 pts) What is the dual of $\mathcal{P}$?

   The dual of $\mathcal{P}$ is the interpretation $\mathcal{P}^*$ in which the points are the lines of $\mathcal{P}$, the lines are the points of $\mathcal{P}$, and incidence is the same as in $\mathcal{P}$.

   (b) (12 pts) Let $\mathcal{P}^*$ be the dual of $\mathcal{P}$. Prove that $\mathcal{P}^*$ is also a projective plane.

   To prove that Axioms I-1, I-2', I-3 and the Elliptic Parallel Property hold in $\mathcal{P}^*$, we need to show that their duals hold in $\mathcal{P}$.
   The dual of Axiom I-1 says that any two distinct lines have a unique point incident with both. The existence part of this statement in the Elliptic Parallel Property, which we know holds in $\mathcal{P}$. The uniqueness part is Proposition 2.1, which holds in any incidence geometry, as we proved in class.
   The dual of Axiom I-2' says that any point has at least three distinct lines incident to it. Let P be a point. By Proposition 2.4, there is a line l which does not pass through P. By Axiom I-2', l has at least three distinct points A, B, and C on it. Clearly, P is not equal to any of these points. By Axiom I-1, there exist unique lines $\overleftrightarrow{PA}$, $\overleftrightarrow{PB}$, and $\overleftrightarrow{PC}$. If any two of these lines were the same, then by Axiom I-1, they would have to be equal to l. But that cannot be since P is incident to these three lines, but not to l. So these are three distinct lines incident with P.
   The dual of Axiom I-3 says that there exist three distinct nonconcurrent lines. This is Proposition 2.2 and we proved it in class.
   Finally, the dual of the Elliptic Parallel Property says the any two distinct points are incident to a common line, which is part of Axiom I-1 in $\mathcal{P}$.

5. (15 pts) Invent a model that satisfies Axioms 1–7 of our axiomatic geometry. First, clearly state your interpretation of the undefined terms (point, line, lie on, between). Then prove that your model satisfies Axioms 1–7.

   Let $S = \{A, B\}$ be a set of two elements. Let points be the elements of S and the only line the set S itself. Let a point lie on a line if it is an element of that line. That is, in our case, both points lie on the only line in our interpretation. Finally, let PQR be always false for any three points P, Q, R. (That our interpretation has only two points does not relieve us of having to define between because a priori, A, B, and C need not be distinct.)

   Axiom 1 holds because A is a point and S is a line. Axiom 2 holds because the only line is S and it contains A and B. Axiom 3 holds because the only distinct points are A and B, and they are on S, which is the only line. Axioms 4–7 hold because they are conditional statements in which the premise can never be true.

   Therefore this interpretation is a model of Axioms 1–7.

6. (15 pts) **Extra credit problem.** Show that Axiom 7 of our Axiomatic Geometry is independent of Axioms 1–6. (Hint: you can do this by inventing two models: one in which Axioms 1–7 hold, and one in which Axioms 1-6 hold but Axiom 7 does not. Your solution to problem 5 will help.)

   First, note that the model in the solution to problem 5 shows that Axiom 7 can hold while Axioms 1–6 hold.

   We will now give a model in which Axioms 1–6 hold, but Axiom 7 does not. Let $S = \{A, B, C\}$ be a set of three elements. Let the points be the elements of S and the only line S itself. Let a point lie on a line if it is an element of that line. That is, in our case, all three points lie on the only line in our interpretation. Finally, let PQR be always false for any three points P, Q, R.

   Axiom 1 holds because A is a point and S is a line. Axiom 2 holds because the only line is S and it contains A and B. Axiom 3 holds because any two distinct points lie on S, which is the only line. Axioms 4–6 hold because they are conditional statements in which the premise can never be true. Finally, Axiom 7 fails to hold, because even though A, B, and C are distinct and collinear, none of ABC, BAC, and ACB are true.