Math 510 Final Exam Solutions

Dec 17, 2010

1. (10 pts) Let $\mathcal{P}$ be a projective plane. Prove that if l and m are are any two distinct lines in $\mathcal{P}$, then there exists a point P in $\mathcal{P}$ that does not lie on either l or m.

   By Axiom I-2', there exist points X_1, X_2 on l and Y_1, Y_2 on m. By Axiom I-1, at least one of X_1 and X_2 is not incident to m, otherwise $m = l$. WLOG $X_1 \not\mathrel{I} m$. Similarly, WLOG $Y_1 \not\mathrel{I} l$. Notice this also implies $X_1 \neq Y_1$. By Axiom I-1, there is a unique line n through X_1, Y_1. By Axiom I-2', there is at least one more point $P \mathrel{I} n$, different from X_1 and Y_1.

   Notice that $P \not\mathrel{I} l$. If P were on l then both l and n would pass through the distinct points X_1 and P, which would make $n = l$. But n also passes through Y_1 and l does not. That is a contradiction. Analogously, $P \not\mathrel{I} m$.

2. (15 pts) In the context of our axiomatic geometry, prove

   Theorem 14. If ABC and $AC \equiv DF$ then there exists a point E such that DEF and $AB \equiv DE$ and $BC \equiv EF$.

   In your proof, you may assume that any result that precedes Theorem 14 in the list has already been proved.

   By Theorem 10, we have P such that PDF. $A \neq B$ and $P \neq D$ because we know ABC and PDF. By Axiom 13 applied to $A \neq B$ and $P \neq D$, there exists E such that PDE and $AB \equiv DE$. Similarly, there exists a point F' such that DEF' and $BC \equiv EF'$. By Axiom 12 with $AB \equiv DE$ and $BC \equiv EF'$, we get $AC \equiv DF'$. By Theorem 4 applied to PDE and DEF', we have PDF'. Now Axiom 14 with PDF, PDF', $AC \equiv DF$, and $AC \equiv DF'$ gives $F = F'$. Hence DEF and $BC \equiv EF$.

3. (15 pts) In the context of our axiomatic geometry, prove

   Theorem 18. If $AB < CD$ then $AB \not\equiv CD$ and $CD \not< AB$.

   In your proof, you may assume that any result that precedes Theorem 18 in the list has already been proved.

   Suppose $AB \equiv CD$. By symmetry, $CD \equiv AB$. By Theorem 16, $AB < CD$ and $CD \equiv AB$ imply $AB < AB$. Then there exists E such that AEB and $AB \equiv AE$. By symmetry, $AE \equiv AB$. But this contradicts Theorem 11 because AEB. Therefore $AB \not\equiv CD$.

   Now suppose $CD < AB$. By Theorem 16, $AB < CD$ and $CD < AB$ imply $AB < AB$. We have seen that this leads to a contradiction, so $CD < AB$ cannot be true either.

4. (10 pts) Suppose that in our axiomatic geometry, we replace Axiom 9 with

   Axiom 9'. If ABC and BCD then ACD and ABD.

   This may of course render any theorems that follow Axiom 9 invalid, since their proofs may rely on the original Axiom 9. Prove that

   Theorem 4'. If ABC and ADC and $B \neq D$, then ABD or ADB.

   follows from Axioms 1–8 and 9'. (This shows that the axioms are to some extent a matter of choice.)

   Suppose ABC and ADC and $B \neq D$. By Theorem 3(d), A, B, C, and D are distinct and collinear. In particular, A, B, D are distinct and collinear. By Axiom 7, ABD or BAD or ADB. Suppose BAD. By Axiom 9', BAD and ADC imply BAC. But BAC contradicts ABC by Theorem 1. Hence either ABD or ADB.

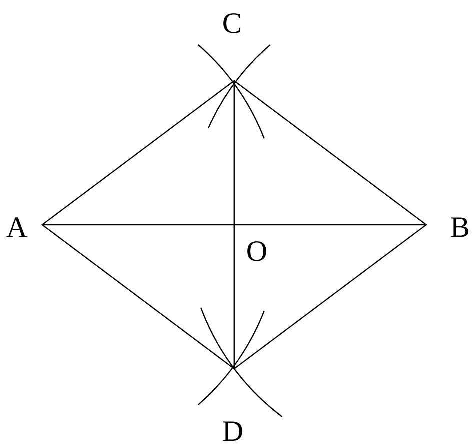
5. (a) (3 pts) Given two angles, define what it means for them to be supplementary.

Two angles $\sphericalangle ABC$ and $\sphericalangle CBD$ are supplementary, if they have one side in common (namely $\overrightarrow{BC}$) and the other two sides are opposite rays.

(b) (3 pts) Define right angle.

An angle is a right angle if it has a supplementary angle that it is congruent to.

(c) (8 pts) Use a straight edge and compass to construct a right angle. Be sure to list the steps of your construction. Use any axioms/theorems from Euclidean geometry to prove that what you constructed indeed satisfies the definition of a right angle.

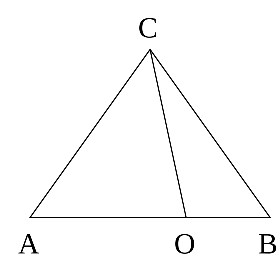


Construction:

1. Choose two distinct points A and B and connect them with a line segment.
2. Draw a circle centered at A with radius more than half of AB. Draw a circle centered at B with the same radius.
3. The two circles intersect in two points C and D. Connect these points with a line segment. Let O be the point where AB and CD intersect. The $\sphericalangle AOC$ is a right angle.

Proof: First notice that $AC \equiv AD \equiv BC \equiv BD$ by construction. Hence $\triangle ACD$ and $\triangle BCD$ are congruent by the Side-Side-Side Theorem. So $\sphericalangle ACO \equiv \sphericalangle BCO$. It follows from this and $AC \equiv BC$ and $CO \equiv CO$ that $\triangle ACO$ and $\triangle BCO$ are congruent by the Side-Angle-Side Theorem. Hence $\sphericalangle AOC \equiv \sphericalangle BOC$. Also, $\sphericalangle AOC$ and $\sphericalangle BOC$ are clearly supplementary. Therefore $\sphericalangle AOC$ is a right angle.

Note: You may be tempted to argue that $\triangle ACB$ is isosceles, hence $\sphericalangle OAC \equiv \sphericalangle OBC$. This and $AC \equiv BC$ and $CO \equiv CO$ imply that $\triangle ACO$ and $\triangle BCO$ are congruent by the Side-Side-Angle Theorem. But there is no such thing as the Side-Side-Angle Theorem. Look at the diagram to the right to see that the statement you would expect in an SSA Theorem is actually false. $\triangle ACO$ and $\triangle BCO$ are obviously not congruent even though $AC \equiv BC$, $CO \equiv CO$, and $\sphericalangle OAC \equiv \sphericalangle OBC$.



6. (a) (6 pts) Let $S = \{A, B, C, D\}$. Consider the following interpretation of incidence geometry. Points are the elements of S. Lines are the 2-element subsets of S (e.g. $\{A, C\}$). A point X is incident to a line l if $X \in l$. Prove that this interpretation is a model of incidence geometry.

First, note that through any two distinct points X and Y in S, there is one and only one line, namely $\{X, Y\}$. Hence Axiom I-1 holds.
Now note that every line in S is of the form $\{X, Y\}$ and therefore is incident to the two distinct points X and Y. Hence Axiom I-2 holds.

Finally, A, B, C are three distinct noncollinear points because there is no line that would contain all three. Hence Axiom I-3 holds.

(b) (10 pts) Construct another 4-point model of incidence geometry which is not isomorphic to the one in part (a). Obviously, you need to prove that your model is not isomoprhic to the one in part (a).

Let $T = \{A', B', C', D'\}$. Let the points be the elements of T and the lines the following subsets of T: $\{A', B', C'\}$, $\{A', D'\}$, $\{B', D'\}$, and $\{C', D'\}$. Let a point be incident to a line if it is contained in it.
First, we will verify that the above interpretation is a model of incidence geometry. Note that through any two distinct points there is a unique line:

$$\begin{aligned} A', B' \, \mathtt{I} \, \{A', B', C'\}, &\qquad A', C' \, \mathtt{I} \, \{A', B', C'\}, \\ A', D' \, \mathtt{I} \, \{A', D'\}, &\qquad B', C' \, \mathtt{I} \, \{A', B', C'\}, \\ B', D' \, \mathtt{I} \, \{B', D'\}, &\qquad C', D' \, \mathtt{I} \, \{C', D'\}. \end{aligned}$$

Hence Axiom I-1 holds.
Now notice that every line contains at least two distinct points, hence every line is incident to at least two distinct points. So Axiom I-2 also holds.
Finally, notice that A', B', D' are three distinct noncollinear points. Hence Axiom I-3 also holds.
But this model cannot be isomorphic to the model in part (a), because isomorphism requires a one-to-one correspondence between lines. But this model has 4 lines whereas the model in part (a) has 6 lines. So no one-to-one correspondence between lines is possible.

7. (15 pts) **Extra credit problem.** In class, we gave an interpretation of our axiomatic geometry on $\mathbb{R}^n$ by defining a point to be an element of $\mathbb{R}^n$, a line between distinct points $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$ to be the set

$$l_{AB} = \{(x_1, \ldots, x_n) \mid x_i = a_i + t(b_i - a_i) \text{ where } t \in \mathbb{R}\},$$

and point X lies on the line l to mean $X \in l$. Then we proved that this interpretation satisfied Axioms 1–3. We then defined between in terms of distance and proved that this satisfied Axioms 4–9. Consider the following definition instead: for three distinct points A, B, C, we say B is between A and C if there exists a real number $0 < t < 1$ such that $b_i = a_i + t(c_i - a_i)$ for all $1 \le i \le n$.

(a) (5 pts) Prove that this definition of betweenness satisfies Axiom 4.

Suppose ABC. Then there exists $0 < t < 1$ such that

$$b_i = a_i + t(c_i - a_i) = (1 - t)a_i + tc_i$$

for all i. Let $s = 1 - t$. Notice that $0 < s < 1$ and

$$b_i = sa_i + (1 - s)c_i = c_i + s(c_i - a_i)$$

for all i. This shows CBA.

(b) (10 pts) Prove that this definition of betweenness satisfies Axiom 5.

Suppose ABC. Then there exists $0 < t < 1$ such that $b_i = a_i + t(c_i - a_i)$ for all i. Now suppose that BAC also holds. Then there exists $0 < s < 1$ such that $a_i = b_i + s(c_i - b_i)$

for all i. Substitute b_i from the first equation into the second:

$$\begin{aligned} a_i = b_i + s(c_i - b_i) &= a_i + t(c_i - a_i) + s(c_i - a_i - t(c_i - a_i)) \\ &= (1 - t - s - st)a_i + (t + s - st)c_i. \end{aligned}$$

Now collect all the terms with a_i in them on one side of the equation and all the terms with c_i on the other side:

$$(t + s - st)a_i = (t + s - st)c_i.$$

If $t + s - st \neq 0$, then we have shown that $a_i = c_i$ for all i, contradicting the fact that $A \neq C$. In fact, since $s, t > 0$,

$$0 < t + s(1 - t) = t + s - st.$$