Lecture notes for Math 510

12/6/10

These notes grew out of Prof. David Whitman's at the San Diego campus of SDSU. I have made some changes, added in some details, expanded some arguments, and added the section titled "Why axiomatic geometry?"

This is the instructor version. The student version has some of the proofs missing.

1. Linear order

We are going to use the following terms without defining them: point, line, lie on (as in a point lies on a line, or a line contains a point), between, congruent. We will instead give a set of axioms that describe their behavior.

We will use capital letters to denote points and lower-case letters to denote lines.

Axiom 1. *There exist at least one point and one line.*

Axiom 2. *Every line contains at least two distinct points.*

Axiom 3. *Any two distinct points have a unique line containing them.*

Let ABC mean that the point B is between the points A and C. Remember that between is an undefined primitive. We did not define what it means here, only introduced a convenient notation for it.

Axiom 4. *If ABC then CBA.*

Axiom 5. *If ABC is true, then BAC is not true.*

We will want to restrict ABC to points that lie on the same line.

Definition 1. We say that two or more points are *collinear* if there is a line that they all lie on.

Note that this definition does not require that the points all be distinct. But for the purposes of betweenness, we will want points to be distinct.

Axiom 6. *If ABC then A, B, and C are distinct and collinear.*

A technical note: all our statements (axioms, theorems, etc) are universally quantified. This means that when we talk about points A, B, C, we really mean any three points. For example, Axiom 4 really means "For any points A, B, C, if ABC then CBA." The reason we have left out the universal quantifiers is that they would make our sentences unnecessarily cumbersome. This has not caused any problems, as we have understood that A, B, C referred to any three points, not three particular points called A, B, C.

We are ready to prove our first theorem. It will be convenient to start with a small lemma.

Lemma 1. *If ABC is false then CBA is false.*

Proof. Axiom 5 applied to CBA gives if CBA then ABC. Now take the contrapositive. □

Theorem 1. *If ABC is true than the following are false: BAC, CAB, ACB, and BCA.*

Proof. The first of these follows directly from Axiom 5.

Now apply Lemma 1 to not BAC to get not CBA.

Using Axiom 4, we get CBA. Applying Axiom 5 to CBA, we find that BCA must be false. Now apply Lemma 1 to not BCA to get not ACB. □

Theorem 2. *Let $A, B \in k$ and $A, B \in l$. If $k \neq l$, then $A = B$.*

Proof. This is for you to prove. □

Axiom 7. *If A, B, and C are distinct and collinear then ABC or BAC or ACB must be true.*

Theorem 3.

(a) *If ABC and ACD, then A, B, C, D are distinct and collinear.*
(b) *If ABC and BCD, then A, B, C, D are distinct and collinear.*
(c) *If ABC and ABD and $C \neq D$, then A, B, C, D are distinct and collinear.*
(d) *If ABC and ADC and $B \neq D$, then A, B, C, D are distinct and collinear.*

Proof. This is for you to prove. □

We will now add two more axioms dealing with the idea of betweenness. Think about what these mean and why we want them to hold. (You may want to draw a picture.) While it may be quite obvious to you that these should be true when you think about lines and points in the Euclidean plane, they do not follow from the axioms we already stated.

Axiom 8. *If ABC and ACD then ABD and BCD.*

Axiom 9. *If ABC and ADC and $B \neq D$, then ABD or ADB.*

Theorem 4. *If ABC and BCD then ACD and ABD.*

Proof. This is for you to prove. □

Theorem 5. *If ABC and ABD and $C \neq D$ then ACD or ADC.*

Proof. This is for you to prove. □

Theorem 6. *If ABC and ABD and $C \neq D$ then BCD or BDC.*

Proof. This is for you to prove. □

Theorem 7. *If ABC, BCD, and ADE, then BCE and ACE.*

Proof. This is for you to prove. □

Definition 2. Let $A \neq B$. The *line segment* $\overline{AB}$ is the set

$$\overline{AB} = \{X \mid X = A \text{ or } X = B \text{ or } AXB\}.$$

Definition 3. How would you define ray?

Theorem 8. *Let $A \neq B, C$. If $B \in \overrightarrow{AC}$ then $\overrightarrow{AB} = \overrightarrow{AC}$.*

Proof. This is for you to prove. □

2. Line segment congruence

In this section, we will set up an axiomatic system to allow us to compare line segments. Recall that we already have already defined what a line segment is (see Definition 2).

Let $\overline{AB} \equiv \overline{CD}$ mean that $\overline{AB}$ and $\overline{CD}$ are congruent. Just like in the previous section, we will not define what congruence means, but we will describe how it behaves with a set of axioms. We will then use these axioms to prove theorems about congruence. To make writing easier, we will usually drop the line from above AB to denote the line segment $\overline{AB}$. This does not usually cause a problem because we have nothing else that we would denote by AB.

First a quick observation about the line segment AB:

Lemma 2. *If $A \neq B$ then $AB = BA$.*

Proof. We proved this in class. □

This has some notable consequences for congruence. For example, if $AB \equiv CD$ then also $BA \equiv CD$, $AB \equiv DC$, and $BA \equiv DC$.

Axiom 10. $AB \equiv AB$

Axiom 11. *If* $AB \equiv CD$ *and* $CD \equiv EF$ *then* $AB \equiv EF$

Axiom 12. *If* ABC, DEF, $AB \equiv DE$, *and* $BC \equiv EF$ *then* $AC \equiv DF$.

Axiom 13. *If* $A \neq B$ *and* $C \neq D$ *then there exists a point* E *such that* CDE *and* $AB \equiv DE$.

Axiom 14. *If* ABC, ABD, *and* $EF \equiv BC$, *and* $EF \equiv BD$ *then* $C = D$.

Theorem 9. *If* ABC, ABD, *and* $BC \equiv BD$ *then* $C = D$.

Proof. We proved this in class. □

Theorem 10. *If* $A \neq B$ *then there exists a point* C *such that* CAB.

Proof. We proved this in class. □

Theorem 11. *If* ABC *then* $AB \not\equiv AC$.

Proof. This is for you to prove. □

Theorem 12. *If* $AB \equiv CD$ *then* $CD \equiv AB$.

Proof. This is for you to prove. □

We now know that segment congruence is symmetric. It will make our subsequent proofs smoother to refer to this property as symmetry, rather then referencing this theorem every time.

Theorem 13. *If* ABC, DEF, $AB \equiv DE$, *and* $AC \equiv DF$ *then* $BC \equiv EF$.

Proof. We proved this in class. □

Theorem 14. *If* ABC *and* $AC \equiv DF$ *then there exists a point* E *such that* DEF *and* $AB \equiv DE$ *and* $BC \equiv EF$.

Proof. This is for you to prove. □

Theorem 15. *If* A, B, C *are distinct and collinear and* $AB \equiv BC$ *then* ABC.

Proof. This is for you to prove. □

3. Comparing line segments

The intuitive idea behind congruence of line segments is that two line segments should be congruent if they are the same size. Of course, in our axiomatic system, we don't really have a notion of size and we cannot measure the length of a line segment. In $\mathbb{R}^n$, we could define the distance of two points, so we could just say that the length of the line segment AB is $d(A, B)$. But this is specific to $\mathbb{R}^n$ and may not be available in another geometric system where our axioms still hold. We will instead use congruence to compare the sizes of line segments, as in the next definition.

Definition 4. We say $AB < CD$ if there exists a point E such that CED and $AB \equiv CE$.

Theorem 16.

(a) *If* $AB < CD$ *and* $CD \equiv EF$ *then* $AB < EF$.
(b) *If* $AB < CD$ *and* $CD < EF$ *then* $AB < EF$.
(c) *If* $AB \equiv CD$ *and* $CD < EF$ *then* $AB < EF$.

Proof. We proved part (a) in class. Parts (b) and (c) are yours to prove. □

Theorem 17. *If AB and CD are segments then $AB < CD$ or $AB \equiv CD$ or $CD < AB$.*

Proof. We proved this in class. □

Theorem 18. *If $AB < CD$ then $AB \not\equiv CD$ and $CD \not< AB$.*

Proof. This is for you to prove. □

Theorem 19. *If ABC, ADC, $AB \equiv BC$, and $AD \equiv DC$ then $B = D$.*

Proof. This is for you to prove. □

4. Why axiomatic geometry?

The motivation for studying geometry through an axiom system as this is that our axioms can fit many different kinds of geometries. Points and lines can mean different things in different geometries. We have seen such things when we constructed and tested models for incidence geometry earlier in the course.

The great thing about axiomatic geometry is that in any interpretation where all of our axioms hold, all of our theorems must also hold because we only relied on the axioms to prove them. This becomes particularly important when you have to work with a geometry that is quite different from the Euclidean geometry we are used to. In fact, there are many such strange geometries and they do play important roles in various branches of mathematics and physics. For example, hyperbolic geometry is central in general relativity, and geometries with ultrametric distances are at the heart of an area of number theory called p-adic analysis. Intuition based on familiar drawings of points and lines does not get you very far in such geometries. If you want to prove theorems, you need to start with the axioms. Playing with the axioms and proving theorems should help you develop an intuition about these other geometries too.

We are now going to give a familiar interpretation of our axiomatic geometry and prove that it is a model.

Recall that

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_1, x_2, \dots, x_n \in \mathbb{R}\}$$

where $(x_1, \dots, x_n)$ is an ordered n-tuple. We call x_i the i-th *coordinate* of $(x_1, \dots, x_n)$. We will call the elements of the set $\mathbb{R}^n$ *points*. We will continue to denote them by capital letters, but we will use lower-case letters for the coordinates. For two points $A = (a_1, \dots, a_n)$ and $B = (b_1, \dots, b_n)$, we say $A = B$ if $a_i = b_i$ for all $i = 1, \dots, n$.

We will now define what a line is. In order to do so, we will need two distinct points. Let $A = (a_1, \dots, a_n)$ and $B = (b_1, \dots, b_n)$ be distinct points in $\mathbb{R}^n$. We say that a *line through A and B* is the set of points

$$l = \{(x_1, \dots, x_n) \mid x_i = a_i + t(b_i - a_i) \text{ where } t \in \mathbb{R}\}.$$

Note that this is the usual equation of a line from analytic geometry. Also, it is clear that $A, B \in l$ with $t = 0$ and $t = 1$. We will see that we can write down other equations of lines that go through A and B. When we are not interested in naming two points that the line goes through, we will simply talk about the line, rather than the line through A and B.

You can now verify that Axioms 1–3 hold.

We need to define our interpretation of betweennes. For three distinct points A, B, C let ABC hold if and only if $d(A, C) = d(A, B) + d(B, C)$, where d is the usual Euclidean distance on $\mathbb{R}^n$:

Definition 5. For $A, B \in \mathbb{R}^n$, the *distance* of A and B is

$$d(A, B) = \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2}.$$

Here are a few useful properties of distance.

Theorem 20. *Let A, B, C be points.*

(a) *Nonnegativity: $d(A,B) \geq 0$ with $d(A,B) = 0$ if and only if $A = B$.*
(b) *Symmetry: $d(A,B) = d(B,A)$.*
(c) *Triangle Inequality: $d(A,C) \leq d(A,B) + d(B,C)$ with equality if and only if $B \in \overline{AC}$.*

You can now verify that Axioms 4 and 5 hold.

We verified Axiom 6 in class. You can now verify Axioms 7–9.

We are ready to define line segment congruence in our interpretation. We will now really need to use distance. We will say $AB \equiv CD$ if $d(A,B) = d(C,D)$.

We verified Axioms 10 and 11 in class. You should prove that Axioms 12–14 also hold.