Math 510 Exam 1 Solutions

Feb 26, 2007

1. (a) (3 pts) State the definition of isometry.

   An isometry is a transformation of a geometric space which preserves distances. (It also preserves angles, but this follows from preserving distances.)

   (b) (3 pts) Give three examples of isometries in the Euclidean plane.

   Reflection across a line, rotation about a point, translation along a line.

   (c) (3 pts) Give three examples of symmetries a line in the Euclidean plane has.

   Reflection across itself, $180°$ rotation about any point on the line, translation along itself.

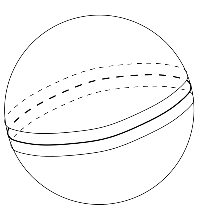
   (d) (6 pts) One of our informal notions of what a straight line should be is "the shortest path between two distinct points." Explain one reason why this would not be a good formal definition.

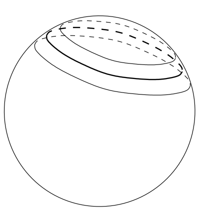
   One problem is that there are infinitely many paths between two distinct points and we cannot measure the lengths of all of them. So how can we decide if the one we are looking at is the shortest?
   Another problem is that the way we measure the length of a path is that we approximate it with short line segments then we take the limit as the length of these line segments goes to 0. (Remember the path length formula from calculus?) So in order to find the length of a path, we already need to know what a straight line is.

2. (a) (3 pts) State the definition of great circle.

   A great circle is a circle on the surface of a sphere which lies in a plane that passes through the center of the sphere.

   (b) (7 pts) We said that on the surface of a sphere, a great circle fits our idea of straightness. Explain why a great circle would look straight to someone whose world is limited to the surface of a sphere. Why would a circle that is not a great circle look curved to someone living in such a spherical world? (Hint: explain how they could detect intrinsic curvature.)





   If you take two parallel circles, one on each side of the great circle equally close to it, then the circumference of these circles is the same. So if you imagine that these two parallel circles correspond to the tire tracks of a car following the great circle, then the tires on the left and right hand sides of the car travel the same distances. Hence the car is not turning. If you do the same thing for a circle that is not a great circle, then one of the two parallel circles will be shorter than the other. Therefore the tires on that side of the car cover less distance. Therefore the car must be turning to that side as it is driving along the original circle.
   Look at the pictures on the left. This kind of intrinsic curvature can be detected even without driving a car. (Think, you don't need to drive a car to tell a curved road from a straight one.)

3. Refer to the set of definitions and axioms on the last page. Using these axioms, prove the following theorems. In this exercise, if you want to refer to a theorem we proved in class or in the lecture notes, you will have to give its complete proof.
   (a) (10 pts) If ABC is true than the following are false: BAC, CAB, ACB, and BCA.

      This proof is in your lecture notes.

   (b) (10 pts) If ABC and BCD, then A, B, C, D are distinct and collinear. (Hint: you may use the result of the previous problem.)

      By Axiom 6, A, B, and C are distinct. By the same axiom applied to BCD, B, C, and D are distinct. Notice that A and D could still be the same. Suppose $A = D$. Then BCD is the same as BCA. But ABC and BCA contradict according to Theorem 1. So $A \neq D$, and A, B, C, D are distinct.
      By definition, if ABC then A, B, and C lie on some line k. Similarly, BCD implies B, C, and D lie on a line l. But k and l both go through the distinct points B and C, hence Axiom 3 shows $k = l$. Therefore A, B, C, D all lie on the same line.

   (c) (10 pts) If ABC and BCD then ABD and ACD. (Hint: you may use the result of the previous problem.)

      This proof is in your lecture notes.

4. (10 pts) **Extra credit problem.** This is a harder problem. Attempt it only when you are done with everything else.

   Prove that if ABC, then $\overline{AC} \subseteq \overrightarrow{AB} \cap \overrightarrow{CB}$. (Hint: you will have to consider several cases.)

   Let $X \in \overline{AC}$. Then $X = A$ or $X = C$ or AXC. Let's look at these cases one by one.

   Case $X = A$**:** By definition, $A \in \overrightarrow{AB}$. We have ABC, so we know CBA by Axiom 4. Hence $A \in \overrightarrow{CB}$. This shows $X \in \overrightarrow{AB} \cap \overrightarrow{CB}$.

   Case $X = C$**:** By definition, $C \in \overrightarrow{CB}$. We have ABC, so $C \in \overrightarrow{AB}$. Hence $X \in \overrightarrow{AB} \cap \overrightarrow{CB}$.

   Case AXC**:** If $X = B$, then by definition $X \in \overrightarrow{AB}$ and $X \in \overrightarrow{CB}$. Hence $X \in \overrightarrow{AB} \cap \overrightarrow{CB}$. If $X \neq B$, then we can use Axiom 9 on ABC and AXC to get ABX or AXB. In either case, $X \in \overrightarrow{AB}$. By Axiom 4 applied to ABC and AXC, we have CBA and CXA. Use Axiom 9 on these to get CBX or CXB. In either case, $X \in \overrightarrow{CB}$. Hence X is in both $\overrightarrow{AB}$ and $\overrightarrow{CB}$. Therefore $X \in \overrightarrow{AB} \cap \overrightarrow{CB}$.

   We have just shown that for any point $X \in \overline{AC}$, $X \in \overrightarrow{AB} \cap \overrightarrow{CB}$. This is exactly what we had to prove.

   Actually, it is also true that $\overline{AC} = \overrightarrow{AB} \cap \overrightarrow{CB}$. Proving $\overrightarrow{AB} \cap \overrightarrow{CB} \subseteq \overline{AC}$ is similar to what we did above, only it involves a few more cases.