MATH 510 EXAM 2 SOLUTIONS

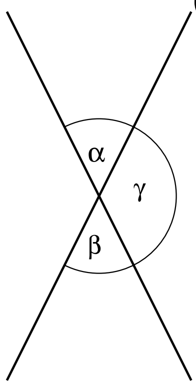
Apr 16, 2007

1. (5 pts each)
   (a) Give one possible definition of angle.

   One possible definition is that it is the union of two different rays with a common endpoint.

   (b) What definition of congruence of angles would be consistent with your definition of angle? Explain carefully why this type of congruence is consistent with your definition of angle.

   One possible definition is that two angles are congruent if there is a sequence of isometries that maps one to the other.



   (c) The Vertical Angle Theorem (VAT) states that opposite angles formed by two distinct intersecting lines are congruent. The following proof of the VAT is often given. (Chances are you learned this proof in high school geometry.)

   The angles α and γ sum to be a straight angle (180°). The angles β and γ also sum to be a straight angle. Hence $\alpha + \gamma = \beta + \gamma$. We get $\alpha = \beta$ by cancelling γ on both sides of the equation.

   Is this proof valid with your definition of angle and angle congruence? Why or why not?

   No, it is not. This definition relies on adding and subtracting angles. But it is not clear that if we start with two congruent angles and to each add an angle such that the added angles are congruent to each other, the resulting angles are going to be congruent. We have the same problem with subtracting angles.
   Notice that the problem is not with angle addition or subtraction per se, but that we do not know if these respect angle congruence.

2. (10 pts) Let d be the usual distance in $\mathbb{R}^n$.

   Definition. For distinct points $A, B, C \in \mathbb{R}^n$, we say B is between A and C (denoted by ABC) if
   $$d(A, C) = d(A, B) + d(B, C).$$

   Suppose we know

   Theorem (Triangle Inequality)**.** *For any three points A, B, C*
   $$d(A, B) \leq d(A, C) + d(B, C)$$
   with equality if and only if ACB or $A = B$ or $B = C$.

   Use this definition and the Triangle Inequality to show that if A, B, C, D are points in $\mathbb{R}^n$ and ABC and ACD then ABD. (This is what we called Axiom 8 in our axiomatic system.)
   Hint: You will have to use the Triangle Inequality twice.

   This proof is on p. 9 of the lecture notes.

3. (15 pts each) Refer to the definitions, axioms, and theorems on the last page. Using these, prove the following theorems. *In this exercise, if you want to refer to a theorem we proved*

in class or in the lecture notes which is not listed on the last page of this exam, you will have to prove it.

(a) If ABC then $AB \not\equiv AC$.

Here is a sketch of the proof: Assume $AB \equiv AC$. Create a point "to the left of" A. Use Axiom 14 to reach a contradiction.

This is Theorem 14 in the lecture notes. While the proof is given there, it uses Theorem 12, which was not listed on the last page of the exam. But Theorem 12 is an easy consequence of Axioms 10 and 14.
Here is a proof that does not refer to Theorem 12:
First note that $A \neq B$ by Axiom 6. By Theorem 13 (here we need $A \neq B$), there exists D such that DAB. Theorem 4 applied to DAB and ABC says DAC. Since $AB \equiv AB$ (by Axiom 10) and $AB \equiv AC$, we can use Axiom 14 with DAB and DAC to show $B = C$. But this cannot be because Axiom 6 says $B \neq C$. Therefore AB and AC cannot be congruent.

(b) If $AB < CD$ and $CD \equiv EF$ then $AB < EF$.

This is Theorem 19.(a) in the lecture notes. The proof is given there.

4. (10 pts) **Extra credit problem.** Attempt this problem only when you are done with everything else. It is not hard, but we have not done in in class or on the HW.

Suppose we did not have Axiom 10. Show that we could prove $AB \equiv AB$ using the remaining axioms of segment congruence (and any axioms or theorems about linear order you may need).

There are a number of ways to do this. Here is one proof.

Since AB is a line segment, $A \neq B$. Use Theorem 13 to get a point C such that CAB. By Axiom 6, $C \neq A$. Use Axiom 13 with $A \neq B$ and $C \neq A$ to get a point B' such that CAB' and $AB \equiv AB'$. Hence $A \neq B'$. Now use Axiom 13 with $A \neq B'$ and $C \neq A$ to get a point B'' such that CAB'' and $AB' \equiv AB''$.

We have $AB \equiv AB'$ and $AB' \equiv AB''$. It follows that $AB \equiv AB''$ by Axiom 11. We now have CAB', CAB'', $AB \equiv AB'$, and $AB \equiv AB''$. By Axiom 14, $B' = B''$. We can substitute this into $AB' \equiv AB''$ to get $AB' \equiv AB'$.

Now, there is nothing special about A and B'. A was arbitrary and we got B' by copying an arbitrary line segment AB. So congruence is reflexive with any line segment, not only AB'.