Math 510 Exam 1

Feb 23, 2009

1. (5 pts) Given a ray $\overrightarrow{AB}$ and a point C, such that C is on the line $\overleftrightarrow{AB}$ and is between A and B, can you prove from the definition of ray and the postulates we have so far that $\overrightarrow{AB} = \overrightarrow{AC}$? Why or why not?

   This is not possible to do. We would need to show that every point of $\overrightarrow{AB}$ is also a point of $\overrightarrow{AC}$ and vice versa. So what are the points on $\overrightarrow{AB}$? By definition, it's A, B, and any point D on the line $\overleftrightarrow{AB}$ such that either D is between A and B or B is between A and D. It is easy to prove that A and B are also points of $\overrightarrow{AC}$. It's those other points D that are the source of trouble. We would have to know that if a point D is between A and B, then it is either between A and C or beyond C, i.e. C is between A and D. We have presently no postulate, common notion, definition, theorem, etc that tells us this is true.

2. (5 pts each) Define the following terms:
   (a) The *midpoint* of a segment AB.

   The *midpoint* of a segment AB is a point M on AB between A and B such that $AM \cong MB$.

   (b) Points A, B, and C are *collinear*.

   Points A, B, and C are *collinear* if there exists a line l such that A, B, and C are on l.

   (c) The *triangle* $\triangle ABC$ formed by three non-collinear points A, B, and C.

   The *triangle* $\triangle ABC$ formed by three non-collinear points A, B, and C is the union of the three line segments AB, AC, and BC.

3. (10 pts) Remember that Euclid's postulates allow you to use an unmarked straightedge and a collapsible compass in geometric constructions. Given a line segment AB, construct the perpendicular bisector of AB. Be sure to describe each step of your construction precisely.

   1. Draw a circle of radius AB centered at A.
   2. Draw a circle of radius AB centered at B.
   3. The above two circles will intersect at two points P and Q. Draw the line $\overleftrightarrow{PQ}$. This is the perpendicular bisector of AB.

4. (5 pts each) State the following definitions:
   (a) Circle of radius OA centered at O.

   The circle of radius OA centered at O is the set of all points P such that $OP \cong OA$.

   (b) Ray $\overrightarrow{AB}$.

   The ray $\overrightarrow{AB}$ is the following set of points: A, B, and any point C on the line $\overleftrightarrow{AB}$ such that either C is between A and B or B is between A and C.

   (c) Right angle.

   An angle is a right angle if it has a supplementary angle it is congruent to.

5. (15 pts) Euclid proves the familiar SAS congruence theorem in Proposition 4 of Book I of his Elements. The statement and proof go like this, slightly adapted to modern language and notation.

   Theorem. *Let $\triangle ABC$ and $\triangle DEF$ be two triangles such that $AB \cong DE$, $AC \cong DF$, and $\sphericalangle BAC \cong \sphericalangle EDF$. Then I say that $BC \cong EF$, $\sphericalangle ABC \cong \sphericalangle DEF$, and $\sphericalangle ACB \cong \sphericalangle DFE$.*

   Proof: If $\triangle ABC$ is overlaid on $\triangle DEF$ so that A is placed on top of D and $\overleftrightarrow{AB}$ is placed on top of $\overleftrightarrow{DE}$, then the point B will coincide with E because $AB \cong DE$. Also, $\overleftrightarrow{AC}$ will coincide with $\overleftrightarrow{DF}$ because $\sphericalangle BAC \cong \sphericalangle EDF$. Therefore the point C will coincide with F because $AC \cong DF$. Since B coincides with E and C coincides with F, the line segments BC and EF also coincide and therefore $BC \cong EF$ by Common Notion 4. Now $\triangle ABC$ and $\triangle DEF$ coincide, therefore $\sphericalangle ABC$ and $\sphericalangle DEF$ coincide and also $\sphericalangle ACB$ and $\sphericalangle DFE$ coincide. Hence $\sphericalangle ABC \cong \sphericalangle DEF$, and $\sphericalangle ACB \cong \sphericalangle DFE$ by Common Notion 4. $\square$

   (a) There is at least one claim in this proof which does not follow from our postulates and common notions. Find this claim and explain why it doesn't follow.

   There are several problems with this proof. One is that the claim that $AB \cong DE$ implies B coincides with E does not follow from any postulate or result we have so far. Similarly, we have no reason to believe that $\sphericalangle BAC \cong \sphericalangle EDF$ implies $\overleftrightarrow{AC}$ coincides with $\overleftrightarrow{DF}$.

   (b) What new postulate would Euclid need to assume so his proof above is correct?

   He would have to assume that if two line segments are congruent then when you overlay them then their endpoints coincide. Similarly, that when two angles are congruent, then when you overlay them, their corresponding sides end up being the same rays. Here is one way to postulate the first of these:
   If A, B, C, D are collinear points such that B is between A and C and B is between A and D and $BC \cong BD$ then $C = D$.

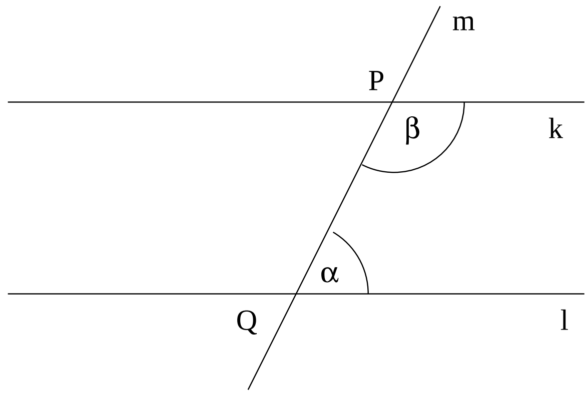
6. (10 pts) **Extra credit problem.** Remember that our statement of the Parallel Postulate is really Playfair's postulate: Given a line l and a point p that does not lie on l, there exists a unique line m through P that is parallel to l.

   Here is Euclid's own formulation: If a line segment intersects two straight lines forming two interior angles on the same side that sum to less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.

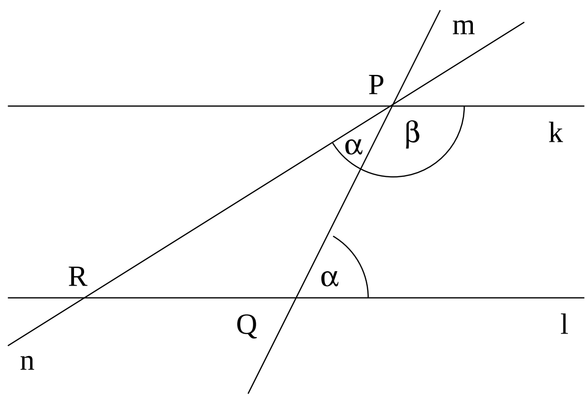
   Find a proof that Playfair's parallel postulate implies Euclid's parallel postulate.

   This is not so easy. The difficulty is not so much coming up with a proof, but being sure that the proof is not circular, that is it only uses results of geometry that are independent of Euclid's parallel postulate.

   Let l, k, m, α, and β be as in the diagram below, such that $\alpha + \beta$ is less than two right angles. (If you have only a line segment instead of m, just extend it to a line.)



Suppose, on the contrary to Euclid's parallel postulate, that l and k do not intersect, that is $l \| m$. Now construct a line n so that the angle between k and n at P is α, as in the diagram below. By Playfair's postulate, n cannot be parallel to l, since m already is. Therefore l and n must intersect at some point R. Now the $\triangle PQR$ has an interior angle α and an exterior angle α too. This violates the Exterior Angle Theorem, which says that any exterior angle must be greater than either of the interior angles not adjacent to it. Fortunately, the proof of this theorem does not use Euclid's parallel postulate.



If you don't recognize this theorem, or are not convinced that it can be proven from only Euclid's first four postulates, here is a proof.

Construct the median RM as below and S such that $RM \cong MS$. Since M is the midpoint of PQ, $PM \cong MQ$. Also, $\sphericalangle RMP$ and $\sphericalangle QMS$ are vertical angles, hence congruent. Now $\triangle RMP \cong \triangle QMS$ by SAS, whose (somewhat deficient) proof you saw earlier in this exam. While we had some concerns about that proof, they weren't that we needed the parallel postulate to prove it.