Math 510 Exam 2 Solutions

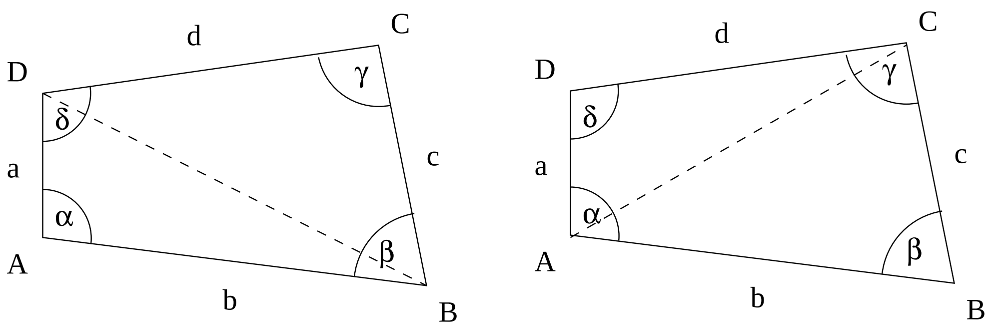
Mar 23, 2009

1. (10 pts) The Egyptians thought that if a quadrilateral had sides of length a, b, c, and d, then its area S was given by the formula $(a+c)(b+d)/4$. Prove that actually

$$4S \leq (a+c)(b+d)$$

with equality holding only for rectangles. (Hint: twice the area of a triangle is $ab\sin(\theta)$, where θ is the angle between sides of length a and b, and $\sin(\theta) \leq 1$, with equality holding only if θ is a right angle.)

Look at the two ways to divide the quadrilateral into two triangles:



The sum of the areas of the four triangles is exactly twice the area of the quadrilateral:

$$2S = \frac{ab\sin(\alpha)}{2} + \frac{bc\sin(\beta)}{2} + \frac{cd\sin(\gamma)}{2} + \frac{ad\sin(\delta)}{2}.$$

Since a, b, c, d are all positive and the sines are all ≤ 1

$$\begin{aligned} 4S &= \underbrace{ab\sin(\alpha)}_{\leq ab} + \underbrace{bc\sin(\beta)}_{\leq bc} + \underbrace{cd\sin(\gamma)}_{\leq cd} + \underbrace{ad\sin(\delta)}_{\leq ad} \\ &\leq ab + bc + cd + ad = (a+c)(b+d) \end{aligned}$$

Equality holds if and only if each of the four sines is 1, that is if all four angles are right angles, that is if the quadrilateral is a rectangle.

2. (10 pts) State the negation of the Euclidean Parallel Postulate. Do this without simply adding "it is not true" or "it is not the case" in front of it, or applying the word not/no directly to any of the quantifiers in it. (Hint: it may help to write down the postulate with logic symbols first.)

The Euclidean Parallel Postulate says that for every line l and every point P not on l there is a unique line m parallel to l through P. In symbols,

$$(\forall l)(\forall P \not\mathrel{\text{I}} l)(\exists! m)(m \parallel l \ \wedge \ P \mathrel{\text{I}} m)$$

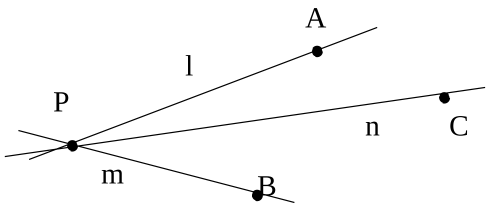
The challenge is to negate $\exists!$. If you think about it, the negation of "there exists a unique" is "either there exists none, or there exist more than one." Here is the negation of the Parallel Postulate in symbols:

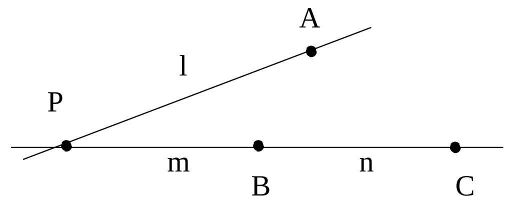
$$(\exists l)(\exists P \not\mathrel{\text{I}} l)\big((\nexists m)(m \parallel l \ \wedge \ P \mathrel{\text{I}} m) \ \vee \ (\exists m, n)(m \neq n \ \wedge m \parallel l \wedge n \parallel l \wedge \ P \mathrel{\text{I}} m \ \wedge \ P \mathrel{\text{I}} n)\big)$$

In words: there exist a line l and a point P not on l such that either there is no line m parallel to l through P or there are two distinct lines m and n parallel to l through P.

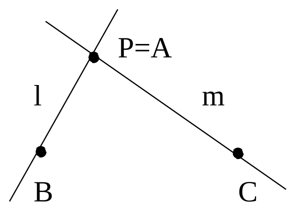
3. (10 pts) Given the Incidence Axioms (listed at the end of this exam), prove that for every point P, there exist at least two distinct lines through P.

By Axiom 3, there exist three distinct non-collinear points A, B, C. If P is not one of A, B, C, then by Axiom 1, there exist unique lines l, m, and n through A and P, B and P, and C and P respectively. If $l = m = n$, then this line is incident to A, B, and C, which would contradict the non-collinearity of A, B, C. Therefore at least two (possibly all three) of l, m, and n are distinct.





If P is one of A, B, C, then without loss of generality, we may assume $P = A$. Now let l and m be the unique lines through P and B and P and C given by Axiom 1. Note that $l \neq m$, otherwise A, B, C would all lie on this line.



4. (a) (10 pts) Describe a model of incidence geometry. Prove that the required axioms hold in your model.

   See Example 1 on p. 72.

   (b) (5 pts) Does the Euclidean Parallel Postulate hold in your model? Does the negation of the Euclidean Parallel Postulate hold in your model? Explain.

   The Euclidean Parallel Postulate does not hold in this model. There are only three lines and any two of them intersect. Hence there are no parallel lines at all.
   If the Euclidean Parallel Postulate is false in this model, then its negation must be true.

5. (15 pts) We have learned that having inconsistent axioms leads to a logical system in which any statement can be proven.

   (a) Make up an inconsistent geometric system by adding a 4th axiom to the three Incidence Axioms. Be sure to prove that your system is now inconsistent.

   An easy way to do this is to add the negation of one of the axioms. For example, we could add the negation of Axiom 3:
   Axiom 4. *Any three distinct points are collinear.*
   Since Axioms 3 and 4 form a contradiction, all four axioms also form a contradiction. Therefore the system is now inconsistent.

   (b) Prove that both the Euclidean Parallel Postulate and its negation follow from your axioms.

   You can do this either using proof by contradiction, or the addition–Modus Tollendo Ponens trick you learned in Math 302. I'll use proof by contradiction.

Suppose that the Parallel Postulate does not hold. By Axiom 3, there exist three distinct, non-collinear points A, B, C. By Axiom 4, A, B, C must be collinear. That's a contradiction. Hence our initial assumption about the Parallel Postulate must be false. Therefore the Parallel Postulate holds.

To prove that the negation of the Parallel Postulate holds, just replace the Parallel Postulate with its negation in the above argument.

6. (10 pts) **Extra credit problem.** Let l and m be parallel lines. If n is a line which intersects l, does it follow that n must also intersect m? If it does, give a proof. If you think the statement is false, disprove it. If you think this statement is independent of the incidence axioms, prove that it is independent.

The statement does not follow from the axioms. Look at the 5-point model of incidence geometry (Example 4 on p. 74-75). Let $l = \{A, B\}$, $m = \{C, D\}$, and $n = \{A, E\}$. Then $l \parallel m$ since they have no point incident to both, n intersects l at point A, but n does not intersect m.

In fact, the statement is always false as long as at least one pair of parallel lines l and m exist. If $n = l$, then n intersects l, but it cannot intersect m because we know $n \parallel m$.

What if there are no parallel lines? Then the statement is true. For it to be false, there would have to exist lines l, m, and n such that $l \parallel m$, n intersects l, and $m \parallel n$. But no such lines can exist, since no parallel lines exist. In fact, there are no parallel lines in the 3-point model of incidence geometry (Example 1 on p. 72). So there is at least one model in which the statement is true.

Since the statement is true in some models and false in others, it must be independent of the incidence axioms.