Math 510 Exam 3 Solutions

Apr 27, 2009

1. (10 pts) In this exercise, you will work in the axiomatic geometry we have been constructing in class. The axioms and theorems are listed at the end of the exam.

   Prove the following statement without using Theorem 8. If ABC, then $\overrightarrow{AC} \subseteq \overrightarrow{AB}$.

   Let $X \in \overrightarrow{AC}$. Then $X = A$ or $X = C$ or AXC or ACX. Let's consider these cases one by one. In each case, we need to show $X \in \overrightarrow{AB}$.

   Case $X = A$**:** In this case, $X \in \overrightarrow{AB}$ by definition.

   Case $X = C$**:** In this case, $X \in \overrightarrow{AB}$ because AXC holds.

   Case AXC**:** There are a few possibilities here. If $X = B$, then $X \in \overrightarrow{AB}$ by definition. If $X \neq B$, we can apply Axiom 9 to AXC and ABC to get AXB or ABX. In either case, $X \in \overrightarrow{AB}$.

   Case ACX**:** Applying Axiom 8 to ABC and ACX gives ABX, hence $X \in \overrightarrow{AB}$.

2. (10 pts each) In this exercise, you will work in an incidence geometry, i.e. one in which the axioms of incidence geometry (listed at the end of this exam) hold.

   Let S be the following statement in the language of incidence geometry: If l and m are two distinct lines, then there exists a point P that does not lie on either l or m.

   (a) Show that S is not a theorem in incidence geometry, i.e. cannot be proved from the axioms of incidence geometry.

   Note that the statement is false in the 3-point model of incidence geometry (Example 1 on p. 72). For example, the lines $\{A, B\}$ and $\{A, C\}$ are distinct, but every point lies on at least one of them.
   Since the statement is false in a model–in which all the axioms hold–it must not be provable from the axioms.

   (b) Show however that the statement holds in every projective plane. Hence $\sim S$ cannot be proved from the axioms of incidence geometry either, so S is independent of those axioms.

   Let l and m be distinct lines in a projective plane. By Axiom I-2', we have at least two distinct points Q, R on l and S, T on m. Note that Q and R cannot both be incident with m, otherwise $l = m$ by the uniqueness part of Axiom I-1. So at least one of Q and R is not on m. Let's say $Q \not{I} m$. Similarly, at least one of S and T is not on l, let's say $S \not{I} l$. Hence Q and S are distinct.
   By Axiom I-1, there is a unique line n through Q and S. By Axiom I-2', there are at least three points on n. At least one of these three points must be different from Q and S. Call this point P. We claim that P does not lie on l or m. Suppose P is on l. Then P and Q are distinct points that lie on both l and n. Hence $l = n$ by Axiom I-1. But this cannot be, since $S I n$ and $S \not{I} l$. The contradiction shows that $P \not{I} l$. By a symmetric argument, $P \not{I} m$. Therefore there is indeed a P that does not lie on either l or m.

3. (15 pts) Construct two nonisomorphic models of incidence geometry with exactly 4 points each. Prove that your models are not isomorphic.

   Let S be a set of 4 elements, say $S = \{A, B, C, D\}$. Consider the following two models.

M_1	M_2
Points: elements of S. Lines: 2-element subsets of S. Incidence: $X \mathbb{I} l$ if $X \in l$.	Points: elements of S. Lines: $\{A, B, C\}, \{A, D\}, \{B, D\}, \{C, D\}$. Incidence: $X \mathbb{I} l$ if $X \in l$.

That M_1 is a model of incidence geometry is proved in the textbook. To see that M_2 is a model, we need to check that the three axioms hold. Indeed, it is easy to check that any two points have a line incident to them, and there is only one such line for each pair of points. E.g. the point B and C have only the line $\{A, B, C\}$ incident to them. Thus Axiom I-1 holds. It is clear that every line has at least two points on it too. Thus Axiom I-2 holds. Finally, there do exist three non-collinear points, e.g. A, B, D. Thus Axiom I-3 holds.

To show that the two models are nonisomorphic, we need to prove that there cannot exist a one-to-one correspondence of points σ and a one-to-one correspondence of lines θ from M_1 to M_2 (or vice versa) which respects incidence. But M_1 has 6 distinct lines while M_2 has only 4. So no one-to-one correspondence can exist between the lines in M_1 and those in M_2.

4. (15 pts) Prove the following statement in the axiomatic geometry we have been constructing in class: If ABC and ADC and $B \neq D$, then BDC or DBC.

If ABC and ADC and $B \neq D$, then A, B, C, and D are distinct and collinear by Theorem 3(d). In particular, B, D, and C are distinct and collinear. By Axiom 7, BDC or DBC or BCD.

Suppose BCD. Then Theorem 4 (applied to ABC and BCD) says ACD and ABD. But ACD contradicts ADC by Theorem 1. Hence BCD must be false.

The only remaining possibility is that BDC or DBC must hold.

5. (15 pts) **Extra credit problem.** Axiom 10 and Axiom 11 in our axiomatic geometry state that line segment congruence is reflexive and transitive. In class we proved that congruence of line segments is also symmetric. Now suppose, that we replace Axiom 10 by the following symmetry axiom:

Axiom 10'. If $AB \equiv CD$ then $CD \equiv AB$.

Can you now prove that congruence is reflexive?

Yes, this can be done. You may be tempted to say that if $AB \equiv CD$ then $CD \equiv AB$, and now by transitivity (Axiom 11) on $AB \equiv CD$ and $CD \equiv AB$, we get $AB \equiv AB$. But what is CD? Let's review again what exactly we are trying to prove: if AB is a line segment, then $AB \equiv AB$. Notice that we can take for granted that we have a line segment AB, otherwise there is nothing to prove. But there is no mention of CD in the statement of the theorem.

Perhaps we can resolve this issue if we start our proof by saying let CD be a line segment such that $AB \equiv CD$. That would be better, but there is still a logical flaw: how do we know that such a line segment exists? We would need to prove that there is in fact a line segment CD in our geometry which is congruent to AB. Where could we get such a line segment? We can use Axiom 13 to construct one!

Here is the complete proof. Let AB be a line segment. Then $A \neq B$ by the definition of line segment. Axiom 13 applied to $A \neq B$ and $A \neq B$ says that there is a point C such that ABC and $AB \equiv BC$. By Axiom 10', $BC \equiv AB$. Now apply Axiom 11 to $AB \equiv BC$ and $BC \equiv AB$ to conclude $AB \equiv AB$.