Lecture notes for Math 510

4/10/09

These notes grew out of Prof. David Whitman's at the San Diego campus of SDSU. I have made some changes, added in some details, expanded some arguments, and added the section titled "Why axiomatic geometry?"

This is the instructor version. The student version has some of the proofs missing.

1. Linear order

We are going to use the following terms without defining them: point, line, between. We will instead give a set of axioms that describe their behavior.

We will use capital letters to denote points and lower-case letters to denote lines.

Axiom 0. *A line is a set of points.*

Axiom 1. *There exist at least one point and one line.*

Axiom 2. *Every line contains at least two distinct points.*

Axiom 3. *Any two distinct points have a unique line containing them.*

Notation: Let ABC denote that the point B is between A and C.

Axiom 4. *If ABC then CBA.*

Axiom 5. *If ABC is true, then BAC is not true.*

Theorem 1. *If ABC is true than the following are false: BAC, CAB, ACB, and BCA.*

Proof. The first of these follows directly from Axiom 5.

Now if CAB were true, then BAC would also have to be true by Axiom 4. But we know BAC is false, so CAB must be false too.

Using Axiom 4, we get CBA. Applying Axiom 5 to CBA, we find that BCA must be false.

Now if ACB were true, then BCA would also have to be true by Axiom 4. But we just proved that BCA is false. Hence ACB must be false too. □

In the process of proving this theorem, we learned the following lemma.

Lemma 1. *If ABC is false then CBA is false.*

Proof. Suppose on the contrary that CBA is true. Then ABC must also be true by Axiom 4. This is a contradiction, so CBA must be false. □

Theorem 2. *Let $A, B \in k$ and $A, B \in l$. If $k \neq l$, then $A = B$.*

Proof. Axiom 3 says that if $A = B$ then there is only one line through them. Hence if $k \neq l$, then A and B must not be distinct. □

Definition 1. We say that three points A, B, and C are *collinear* if there is a line that contains all three.

Note that this definition does not require that the three points be distinct.

Axiom 6. *If ABC then A, B, and C are distinct and collinear.*

You may have noticed that we already know from the definition that if ABC then A, B, and C are collinear.

Axiom 7. *If A, B, and C are distinct and collinear then ABC or BAC or ACB must be true.*