Math 521B Exam 1 Solutions

Feb 25, 2008

1. (10 pts) Prove that $(1\,2\,3\,4)$ is not a product of 3-cycles.

   Note that a 3-cycle $(a\,b\,c)$ can be written as a product of two transpositions. For example, $(a\,b\,c) = (a\,b)(b\,c)$. Hence any 3-cycle is even. On the other hand, $(1\,2\,3\,4) = (1\,2)(2\,3)(3\,4)$, which shows it is odd. Any product of even permutations must be even, hence $(1\,2\,3\,4)$ cannot be a product of 3-cycles.

2. (10 pts) Let $H = \{\beta \in S_5 \mid \beta(1) = 1 \text{ and } \beta(3) = 3\}$. Prove that H is a subgroup of S_5. Is your argument valid when 5 is replaced by any $n \geq 3$?

   Clearly $H \subseteq S_5$. Note $\mathrm{Id} \in H$ as $\mathrm{Id}(1) = 1$ and $\mathrm{Id}(3) = 3$. Suppose $\alpha, \beta \in H$. Then
   $$\alpha\beta(1) = \alpha(\beta(1)) = \alpha(1) = 1.$$
   Similarly, $\alpha\beta(3) = 3$. Hence $\alpha\beta \in H$. Also,
   $$\alpha^{-1}(1) = \alpha^{-1}(\alpha(1)) = \mathrm{Id}(1) = 1.$$
   Similarly, $\alpha^{-1}(3) = 3$. Therefore H is a subgroup of S_5 by the Two-Step Subgroup Test.

   The above argument works fine with any other $n \geq 3$ as it uses nothing special about S_5.

3. (10 pts) If $\beta \in S_7$ and $|\beta^3| = 7$, prove that $|\beta| = 7$.

   Let $n = |\beta|$. Since $|\beta^3| = 7$, we know $\beta^{21} = ()$. Therefore $n|21$. Clearly $n \neq 1, 3$, otherwise $\beta^3 = ()$ and $|\beta^3|$ would be 1. But n cannot be 21 either. To have order 21, the least common multiple of the lengths of the disjoint cycles of β would have to be 21. Therefore at least one of them would have be divisible by 7, and one of them (possibly the same one) would have be divisible by 3. This is impossible since the sum of the lengths cannot be more than 7 in S_7. Thus $|\beta| = 7$.

4. (10 pts) Prove that every element of S_n $(n \geq 2)$ can be written as a product of transpositions.

   First, let $\sigma = (x_1\ x_2\ \ldots\ x_k)$ be a cycle of length at least 2. Then
   $$\sigma = (x_1\ x_2)(x_2\ x_3)\cdots(x_{k-1}\ x_k).$$
   You can verify this by direct computation.

   Now for a general non-identity element σ, just write it first as a product of disjoint cycles (we proved in class this can always be done), then write each cycle as a product of transpositions as above. This gives σ as a product of transpositions.

   If $\sigma = ()$, you can think of it a product of 0 transpositions, or if you don't like that, write $() = (1\,2)(1\,2)$.

   NB: This proof does not actually use that the cycles are disjoint. The only reason to write σ as a product of disjoint cycles first is because we have a theorem that says this is always possible. In fact, any cycles would be good enough for our purpose here.

5. (10 pts)
   (a) Write
   $$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 7 & 5 & 1 & 3 & 2 & 6 \end{pmatrix}$$

in disjoint cycle notation.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 7 & 5 & 1 & 3 & 2 & 6 \end{pmatrix} = (1\,4)(2\,7\,6)(3\,5).$$

(b) Write $(1\,6\,4\,3)(6\,2\,5\,3)(1\,5)$ as a product of disjoint cycles.

$$(1\,6\,4\,3)(6\,2\,5\,3)(1\,5) = (2\,5\,6)(3\,4).$$

(c) Write $(1\,6\,4\,3)(6\,2\,5\,3)(1\,5)$ as a product of transpositions. Is it even or odd?

One possibility is to use the same idea as in problem 4 on each cycle:

$$(1\,6\,4\,3)(6\,2\,5\,3)(1\,5) = (1\,6)(6\,4)(4\,3)(6\,2)(2\,5)(5\,3)(1\,5).$$

This has 7 transpositions, so the permutation is odd.

(d) If $\sigma = (1\,6\,4\,3)(6\,2\,5\,3)(1\,5)$, what is σ^{2007}?

We use the fact that disjoint cycles commute and that the order of an n-cycle is n:

$$\sigma^{2007} = ((2\,5\,6)(3\,4))^{2007} = (2\,5\,6)^{2007}(3\,4)^{2007} = ()(3\,4) = (3\,4).$$

6. (10 pts) Find the number of elements of order 6 in S_5.

To have order 6, the lengths of the disjoint cycles of the permutation must have 6 for the their least common multiple. Their sum cannot be more than 5 either. So the only possibility is that there are two disjoint cycles, one of length 3 and another of length 2. That is $\sigma = (a\,b\,c)(d\,e)$. There are 5 choices for a, 4 for b, 3 for c, 2 for d, and only 1 for e. This would give us 5! choices. But $(a\,b\,c) = (b\,c\,a) = (c\,a\,b)$ and $(d\,e) = (e\,d)$, so 6 different choices will give the same permutation. Hence the number of elements of order 6 in S_5 is $5!/6 = 20$.

7. (15 pts) **Extra credit problem.**

(a) Let $\alpha, \beta \in S_n$ such that $\alpha = (a_1\, a_2\, \ldots\, a_k)$. Prove that

$$\beta\alpha\beta^{-1} = (\beta(a_1)\ \beta(a_2)\ \ldots\ \beta(a_k)).$$

Let $x \in \{1, 2, \ldots, n\}$. If $x = \beta(a_i)$ for some $1 \le i < k$. Then

$$\beta\alpha\beta^{-1}(x) = \beta\alpha\beta^{-1}(\beta(a_i)) = \beta\alpha(a_i) = \beta(a_{i+1}).$$

which is exactly where the cycle $(\beta(a_1)\ \beta(a_2)\ \ldots\ \beta(a_k))$ maps x.
The same argument shows that

$$\beta\alpha\beta^{-1}(\beta(a_k)) = \beta(a_1)$$

as it should be.
Finally, let $x \notin \{\beta(a_1), \beta(a_2), \ldots, \beta(a_k)\}$. Then $\beta^{-1}(x) \notin \{a_1, a_2, \ldots, a_k\}$. Therefore α leaves $\beta^{-1}(x)$ fixed. That is $\alpha(\beta^{-1}(x)) = \beta^{-1}(x)$. So

$$\beta\alpha\beta^{-1}(x) = \beta(\beta^{-1}(x)) = x.$$

In this case, the cycle $(\beta(a_1)\ \beta(a_2)\ \ldots\ \beta(a_k))$ would also leave x fixed.
So $\beta\alpha\beta^{-1}$ permutes the elements of $\{1, 2, \ldots, n\}$ exactly the same way as the cycle $(\beta(a_1)\ \beta(a_2)\ \ldots\ \beta(a_k))$ does. Therefore they must be equal.

(b) Use your result from part (a) to write $(2\,1\,5)(3\,1\,2\,4)(2\,5\,1)$ as a single cycle.

Note that $(2\,5\,1) = (2\,1\,5)^{-1}$ and $(2\,5\,1)$ sends 1 to 2 and 2 to 5. So

$$(2\,1\,5)(3\,1\,2\,4)(2\,5\,1) = (2\,1\,5)(3\,1\,2\,4)(2\,1\,5)^{-1} = (3\,5\,1\,4)$$

by the result in part (a).

(c) Use your result from part (a) to prove the following theorem: If $\alpha, \beta \in S_n$, then $\beta\alpha\beta^{-1}$ is the permutation we get by writing down α in cycle notation and applying β to each entry. (Note this is more general than the statement in part (a) because we are not assuming α to be a cycle.)

Let $\alpha \in S_n$. Suppose $\alpha = \alpha_1\alpha_2\cdots\alpha_k$ where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are cycles. Then

$$\begin{aligned}\beta\alpha\beta^{-1} &= \beta\alpha_1\alpha_2\cdots\alpha_k\beta^{-1}\\ &= \beta\alpha_1\underbrace{(\beta^{-1}\beta)}_{\text{Id}}\alpha_2\underbrace{(\beta^{-1}\beta)}_{\text{Id}}\cdots\underbrace{(\beta^{-1}\beta)}_{\text{Id}}\alpha_k\beta^{-1}\\ &= (\beta\alpha_1\beta^{-1})(\beta\alpha_2\beta^{-1})\cdots(\beta\alpha_k\beta^{-1})\end{aligned}$$

By part (a), $(\beta\alpha_i\beta^{-1})$ is just the cycle α_i with β applied to each entry. This shows that $\beta\alpha\beta^{-1}$ is α with β applied to each entry in each cycle of α.