Math 521B Exam 3 Solutions

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1. (10 pts) Let a and b be non-identity elements of different orders in a group G of order 155. Prove that the only subgroup of G that contains a and b is G itself.

   Here is the first of two slightly different proofs:
   By Lagrange's Theorem, $|a|$ and $|b|$ divide $|G|$. Since a and b are non-identity elements, $|a|, |b| > 1$. So the orders of a and b are 5, 31, or 155.
   Let H be any subgroup that contains a and b. Obviously, $|H| \leq |G| = 155$. If a or b has order 155, then $155 \mid |H|$, since the order of an element divides the order of the group. Therefore $|H| = 155$ and $H = G$.
   Otherwise, $|a| = 5$, $|b| = 31$ or $|a| = 31$, $|b| = 5$. So $5, 31 \mid |H|$. Hence $|H|$ is divisible by $\gcd(5, 31) = 155$. Therefore $|H| = 155$ and $H = G$.

   Here is the other:
   Let H be a subgroup such that $a, b \in H$. By Lagrange's theorem, $|H| \mid |G|$, so $|H|$ is 1, 5, 31, or 155. But $|H| \neq 1$ because H contains non-identity elements a and b.
   If $|H| = 5$, then any non-identity element in H must have order 5. This contradicts the requirement that a and b have different orders.
   The same argument shows that $|H| \neq 31$. Hence $|H| = 155$ is the only remaining possibility and then $H = G$.

2. (10 pts) Prove that every subgroup of D_n whose order is odd is cyclic.

   Remember that the two kinds of elements in D_n are rotations and reflections.
   Let H be such a subgroup. Since $|H|$ is odd, H cannot contain an element of order 2. But every reflection has order 2. So H can only contain rotations. Therefore H is a subgroup of C_n, which is a cyclic group. Any subgroup of a cyclic group is cyclic, so H must be cyclic.

3. (10 pts) If H is a normal subgroup of G and $|H| = 2$, prove that H is contained in the center of G.

   So H must contain two elements. One of these is the identity e. Let us call the other one h. Since e obviously commutes with everything in G, we need to show that h does too.
   Let $g \in G$. Then $gH = Hg$ by normality. Hence $gh \in gH = Hg = \{g, hg\}$. If $gh = g$ then $h = e$ by cancellation. But we know $h \neq e$. So $gh = hg$. This is true for any g, hence h is central.

4. (15 pts)
   (a) Let G be a group and H a subgroup of G. For $a, b \in G$, define $a \sim b$ if $Ha = Hb$. Prove that $\sim$ is an equivalence relation on G.

      Reflexivity: Obviously, $Ha = Ha$ for all a, hence $a \sim a$.

      Symmetry: Suppose $a \sim b$. Then $Ha = Hb$. Hence $Hb = Ha$ and then $b \sim a$.

      Transitivity: Suppose $a \sim b$ and $b \sim c$. Then $Ha = Hb$ and $Hb = Hc$. Hence $Ha = Hc$ and then $a \sim c$.

      Notice that this all comes down to the fact that equality of sets is an equivalence relation.

(b) Show that the right cosets of H form a partition of G.

We will do this by showing that the right cosets are exactly the equivalence classes of $\sim$. Since we know that equivalence classes partition the set, this will do the job.
Let $a \in G$. Its equivalence class under $\sim$ is $[a] = \{b \in G | b \sim a\}$. We want to show $[a] = Ha$.
Let $b \in [a]$. Then $b \sim a$, so $Hb = Ha$. Since $b \in Hb$, $b \in Ha$. Therefore $[a] \subseteq Ha$.
Now, let $b \in Ha$. Then $b = ha$ for some $h \in H$. We showed in class that left multiplication by a fixed element h is a permutation $H \to H$. By the same logic, right multiplication by h is also a permutation $H \to H$. Therefore $Hh = H$. Hence

$$Hb = H(ha) = (Hh)a = Ha.$$

This shows $b \sim a$ and $b \in [a]$. So $Ha \subseteq [a]$.

5. (15 pts)
   (a) Let G be a group and $H \subseteq G$ a subgroup. Prove that if $[G : H] = 2$, then H is normal in G.

   If $[G : H] = 2$ then H has exactly two left cosets and two right cosets. One of the left cosets is H itself. Since the cosets partition G, the other left coset must be $G \setminus H$. The same thing can be said about the right cosets. So the two left and the right cosets are the same: H and $G \setminus H$. Therefore H is normal.

   (b) Is the statement in part (a) true if 2 is replaced by p, where p is any prime? That is, if $[G : H] = p$ for some prime p, does H have to be normal in G?

   This is not true. For example, $[S_3 : \{(), (1\,2)\}] = 3$, which is prime. But we saw in class that $\{(), (1\,2)\}$ is not normal in S_3.

6. (15 pts) **Extra credit problem.** In this problem, you will explore an alternate way to define quotient groups.
   Let G be a group. For two subsets $S, T \subseteq G$, define the product of S and T as

$$ST = \{st \mid s \in S, t \in T\}.$$

   (a) Now, let H be a subgroup of G and K and N two left cosets of H in G. We can multiply K and N as sets using the above definition. Notice that here we don't have to worry if multiplication is well defined. But the result may not be a left coset.
   Prove that if H is normal in G and K and N are left cosets of H, then KN is also a left coset of H.

   Since K and N are left cosets, $K = aH$ and $N = bH$ for some $a, b \in G$. First observe that

$$HH = \{xy \mid x \in H, y \in H\} \subseteq H$$

   by closure. Since $e \in H$, $eH \subseteq HH$. But $eH = H$. Therefore $HH = H$.
   If K and N are left cosets, then $K = aH$ and $N = bH$ for some $a, b \in G$. Now

$$KN = (aH)(bH) = (aH)(Hb) = a\underbrace{HH}_{H}b = aHb = a(Hb) = a(bH) = (ab)H,$$

   which is indeed a left coset of H.

Notice that the result of multiplying aH and bH is exactly what you would expect. In fact, this definition gives the same multiplication on the cosets as the definition we gave in class.

(b) Let $G = S_3$ and $H = \{(), (1\,2)\}$. Show that the product of two left cosets of H need not be a left coset. (Actually, if you prefer to do this with right cosets instead, feel free.)

Take for example the left cosets $K = (1\,3)H = \{(1\,3), (1\,2\,3)\}$ and $N = (2\,3)H = \{(2\,3), (1\,3\,2)\}$. Then

$$\begin{aligned} KN &= \{(1\,3), (1\,2\,3)\}\{(2\,3), (1\,3\,2)\} \\ &= \{(1\,3\,2), (2\,3), (1\,2), ()\}, \end{aligned}$$

which obviously cannot be a left coset of H because it has two many elements.