Math 521B Final Exam Solutions

May 19, 2010

1. (10 pts) Let G be a finite group and let H and K be subgroups of G. Prove that
$$|HK| = \frac{|H||K|}{|H \cap K|}.$$

Since H and K are finite, so is $H \cap K$ and we can let $n = |H \cap K|$. Define the map $\phi : H \times K \to HK$ by $\phi(h, k) = hk$. Since
$$HK = \{hk | h \in H, k \in K\}$$
ϕ is onto. Now let $\sim$ be the relation on $H \times K$ defined by $(h, k) \sim (h', k')$ if and only if $\phi(h, k) = \phi(h', k')$. By Theorem 2.2.7, ϕ induces a one-to-one correspondence between the set of equivalence classes $(H \times K)/\sim$ and HK. We will now show that each equivalence class has exactly $n = |H \cap K|$ elements.

Let $(h, k) \in H \times K$. We will prove that the equivalence class $[(h, k)]$ has n elements by constructing a one-to-one correspondence $\phi : H \cap K \to [(h, k)]$. For $x \in H \cap K$, define $\phi(x) = (hx, x^{-1}k)$. First, we verify that $\phi(x) \in [(h, k)]$. Since $x \in H \cap K$, $hx \in H$ and $x^{-1}k \in K$. So $\phi(x) \in H \times K$. Notice
$$(hx)(x^{-1}k) = h(xx^{-1})k = hk \implies \phi(x) \sim (h, k) \implies \phi(x) \in [(h, k)].$$
Now define $\sigma : [(h, k)] \to H \cap K$ by $\sigma(h', k') = h^{-1}h'$. Notice
$$\sigma\phi(x) = \sigma(hx, x^{-1}k) = h^{-1}hx = x$$
and
$$\phi\sigma(h', k') = \phi(h^{-1}h') = (h(h^{-1}h'), (h^{-1}h')^{-1}k) = (h', (h')^{-1}hk).$$
But
$$(h', k') \in [(h, k)] \implies (h, k) \sim (h', k') \implies hk = h'k' \implies (h')^{-1}hk = h'^{-1}h'k' = k'.$$
So $\phi\sigma(h', k') = (h', k')$. This shows ϕ and σ are inverses, so ϕ is a one-to-one correspondence and $|[(h, k)]| = |H \cap K| = n$.

Now, every equivalence class in $H \times K$ has n elements, so there are
$$\frac{|H \times K|}{n} = \frac{|H||K|}{|H \cap K|}$$
equivalence classes and consequently the same number of elements in HK.

2. (10 pts) Let G be any group and let a be a fixed element of G. Define $\phi_a : G \to G$ by $\phi_a(x) = axa^{-1}$, for all $x \in G$. Show that ϕ_a is an isomorphism.

First, notice that for all $x, y \in G$,
$$\phi_a(x)\phi_a(y) = axa^{-1}aya^{-1} = axya^{-1} = \phi_a(xy).$$
So ϕ_a is a homomorphism. Now for all $x \in G$,
$$\phi_a\phi_{a^{-1}}(x) = \phi_a(a^{-1}xa) = aa^{-1}xaa^{-1} = x$$
and
$$\phi_{a^{-1}}\phi_a(x) = \phi_{a^{-1}}(axa^{-1}) = a^{-1}axa^{-1}a = x.$$
Hence $\phi_a\phi_{a^{-1}} = \phi_{a^{-1}}\phi_a = 1_G$. So ϕ_a has an inverse, and hence ϕ_a is an isomorphism.

3. (10 pts) Prove that any finite cyclic group with more than two elements has an even number of distinct generators.

   Let G be a finite cyclic group such that $|G| > 2$. Remember that $g \in G$ is a generator of G iff $|g| = |G|$. Also, recall that $|g| = |g^{-1}|$. So g is a generator iff g^{-1} is a generator. Now if g is a generator, $|g| = |G| > 2$, so $g \neq g^{-1}$. Therefore the generators of G can be listed as pairs (g, g^{-1}), which shows that there are an even number of them.

4. (10 pts) Prove that the intersection of two normal subgroups is a normal subgroup.

   First, we know from the HW (3.2.17) that the intersection of any number of subgroups is a subgroup. So all we need to show is that if N_1 and N_2 are normal in G, then $N_1 \cap N_2$ is normal too. Let $n \in N_1 \cap N_2$ and $g \in G$. Since N_1 is normal,

$$n \in N_1 \implies gng^{-1} \in N_1.$$

   Similarly, $gng^{-1} \in N_2$. Hence $gng^{-1} \in N_1 \cap N_2$. This shows $N_1 \cap N_2$ is normal.

5. Let G be a group and H a subgroup.
   (a) (3 pts) Define what the left cosets of H are.

   See Definition 3.8.3.

   (b) (10 pts) Let S be the set of left cosets of H. Prove that multiplication on S given by $(aH)(bH) = abH$ is well defined if and only if H is normal.

   Suppose H is normal and $a, b, c, d \in G$ are such that $aH = cH$ and $bH = dH$. We need to show that

$$abH = (aH)(bH) = (cH)(dH) = cdH.$$

   We proved in class that $aH = cH$ iff $c^{-1}a \in H$ and $bH = dH$ iff $d^{-1}b \in H$. Hence

$$(cd)^{-1}(ab) = d^{-1}c^{-1}ab = d^{-1}hb$$

   for some $h \in H$. Also

$$d^{-1}hb = d^{-1}bb^{-1}hb = d^{-1}bh'$$

   where $h' = b^{-1}hb \in H$ by the normality of H. Finally, since $d^{-1}b \in H$ and $h' \in H$,

$$(cd)^{-1}(ab) = d^{-1}bh' \in H.$$

   Hence $abH = cdH$.
   Conversely, suppose $(aH)(bH) = abH$ is well defined. We need to prove that for any $h \in H$ and any $g \in G$, $ghg^{-1} \in H$. So let $h \in H$ and $g \in G$. Notice that

$$\big((gH)(hH)\big)(g^{-1}H) = (ghH)(g^{-1}H) = ghg^{-1}H$$

   (We used parentheses here because we have not yet proved that the multiplication of G/H is associative.) But since $h \in H$, multiplying the elements of H by h only permutes them, so $hH = eH$. Hence

$$\big((gH)(hH)\big)(g^{-1}H) = \big((gH)(eH)\big)(g^{-1}H) = (geH)(g^{-1}H) = gg^{-1}H = H$$

   So $ghg^{-1}H = H$. In particular, $e \in H$, so $ghg^{-1} = ghg^{-1}e \in ghg^{-1}H = H$, which is what we wanted to show.

(c) (10 pts) Let G be a group and N a normal subgroup. Show that the factor set G/N of left cosets of N is a group under the multiplication defined in part (b).

First, let $xN, yN \in G/N$. Then $x, y \in G$, so $xy \in G$. Hence

$$(xN)(yN) = xyN \in G/N.$$

Therefore G/N is closed.
If $xN \in G/N$, then

$$(eN)(xN) = exN = xN$$

and

$$(xN)(eN) = xeN = xN.$$

So $eN \operatorname{Inn} G/N$ is an identity.
Let $xN, yN, zN \in G/N$. Then $x, y, z \in G$. So $(xy)z = x(yz)$ and

$$\big((xN)(yN)\big)(zN) = (xyN)(zN) = (xy)zN = x(yz)N = (xN)(yzN) = (xN)\big((yN)(zN)\big).$$

Finally, if $xN \in G/N$ then $x \in G$, so $x^{-1} \in G$, so $x^{-1}N \in G/N$ and

$$(xN)(x^{-1}N) = xx^{-1}N = eN$$
$$(x^{-1}N)(xN) = x^{-1}xN = eN.$$

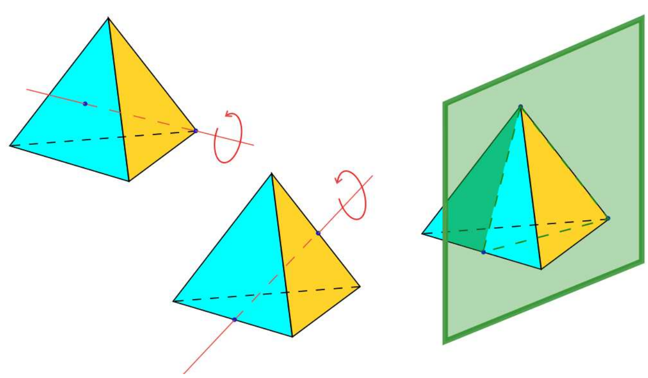
So $x^{-1}N$ is the inverse of xN in G/N.

6. (a) (2 pts) What's a permutation group?

See Definition 3.6.1.

(b) (5 pts) Find all symmetries of a regular tetrahedron.

Label the vertices the tetrahedron by 1, 2, 3, 4. The symmetries of the tetrahedron permute these four vertices, hence we can write them in terms of permutations and they form a subgroup of S_4 and. There are rotational symmetries by 120° and 240° about the four axes that pass through a vertex and the center of the opposite face. There are also rotational symmetries by 180° about the three axes that pass through the midpoints of two opposite edges. There is of course the identity. There are six reflections across the planes that pass through an edge and the midpoint of the opposite edge. That is 18 symmetries so far. The order of S_4 is 24. By Lagrange's theorem, we know that S_4 cannot have a subgroup order 18. In fact, if $H \subseteq S_4$ is a subgroup which contains 18 elements, then it must contain the remaining 6 elements too. So the group of symmetries is all of S_4. The remaining 6 symmetries are reflections composed with rotations. In terms of permutations, these are the 4-cycles in S_4.



Here is an alternate argument. By the labeling above, the symmetries of the tetrahedron form a subgroup of S_4. We will show that this subgroup is all of S_4. Notice that the tetrahedron has a mirror symmetry across the plane through any one edge and the midpoint of the opposite edge. These six reflections permute the vertices as the transpositions $(1\,2)$, $(1\,3)$, $(1\,4)$, $(2\,3)$, $(2\,4)$, $(3\,4)$. Since any permutation in S_4 can be written as a product of transpositions, the products of these six transpositions make up all of S_4. Hence the group of symmetries of the tetrahedron is S_4.

(c) (10 pts) Prove that every group is isomorphic to a permutation group.

See Theorem 3.6.2.

7. (10 pts each) **Extra credit problem.** Let $\phi : G_1 \to G_2$ be a group homomorphism and H_2 a subgroup of G_2. Define the pullback of H_2 to be the set

$$\phi^{-1}(H_2) = \{g \in G_1 \mid \phi(g) \in H_2\}$$

(Note that the notation ϕ^{-1} does not imply that ϕ has an inverse.)

(a) Show that $\phi^{-1}(H_2)$ is a subgroup of G_1.

See Proposition 3.7.6(b).

(b) If N_2 is a normal subgroup of G_2, must $N_1 = \phi^{-1}(N_2)$ be a normal subgroup of G_1?

Yes. See Proposition 3.7.6(b).